HW 5: 5.5, 5.11, 5.13, 5.15, Two-tank problem

5.5 Equivalent bulk modulus of gas trapped in liquid

One key here is to realize the total volume V_T is composed of a volume of gas V_g and a volume of liquid V_L,

$$V_T = V_g + V_L$$

and the definition of equivalent bulk modulus is based upon the total volume (evaluated at the reference pressure and temperature).

$$\beta_e = -\frac{\partial P}{\partial \frac{V_T}{V_{T_0}}}$$

The bulk modulus for the gas is based upon the volume of the gas.

$$\beta_g = -\frac{\partial P}{\partial \frac{V_g}{V_{g_0}}}$$

The bulk modulus for the liquid is based upon the volume of the liquid.

$$\beta_L = -\frac{\partial P}{\partial \frac{V_L}{V_{L_0}}}$$

It is easier to work with the inverse of the bulk modulus definition since we are really working with the volumes.

$$\frac{1}{\beta_e} = -\frac{\partial \frac{V_T}{V_{T_0}}}{\partial P} = -\frac{\partial \frac{(V_g + V_L)}{V_{T_0}}}{\partial P} = -\frac{1}{V_{T_0}} \frac{\partial (V_g + V_L)}{\partial P}$$

Using the volume ratio $x = V_{g0}/V_{T0}$ as the initial volume of gas to the total, we can express the above equation as follows.

$$\frac{1}{\beta_e} = -\frac{1}{V_{T0}} \left[\frac{\partial V_g}{\partial P} + \frac{\partial V_L}{\partial P} \right] = -\frac{1}{V_{T0}} \left[\frac{V_{g0}}{V_{g0}} \frac{\partial V_g}{\partial P} + \frac{V_{L0}}{V_{L0}} \frac{\partial V_L}{\partial P} \right] = \left[\frac{V_{g0}}{V_{T0}} \frac{1}{\beta_g} + \frac{V_{L0}}{V_{T0}} \frac{1}{\beta_L} \right]$$

$$\frac{1}{\beta_e} = \left[\frac{x}{\beta_g} + \frac{1-x}{\beta_L} \right]$$

Inverting the above equation and normalizing with respect to β_L yields the following.

$$\frac{\beta_e}{\beta_L} = \frac{1}{x\frac{\beta_L}{\beta_g} + (1-x)} = \frac{1}{1 + x\left(\frac{\beta_L}{\beta_g} - 1\right)}$$

Now, since the bulk modulus of the liquid is much larger than the gas ($\beta_L / \beta_g >> 1$), the above expression simplifies to the following. Notice that the equivalent bulk modulus is reduced by the presence of any volume of gas.

$$\frac{\beta_e}{\beta_L} = \frac{1}{1 + x\frac{\beta_L}{\beta_g}}$$

Further, using the bulk modulus for a gas as nP, we get the following equation.

$$\frac{\beta_e}{\beta_L} = \frac{1}{1 + x\frac{\beta_L}{nP_{absolute}}}$$

The equivalent bulk modulus will be reduced to one half its original value if the second term in the denominator is equal to 1. Using a value of bulk modulus of the liquid of 1.4 x 10^6 kPa and the gas is air at 100 kPa at high frequency (which means that $n = 1.4$), then the volume ratio is given as follows.

$$x = \frac{n\,P_{absolute}}{\beta_L} = \frac{1.4 \times 100\text{ kPa}}{1.4 \times 10^6\text{ kPa}} = 0.0001 = 0.01\%$$

This illustrates how sensitive the response of a hydraulic system is to the presence of any air. A tiny bubble in a hydraulic system can reduce the bulk modulus which will affect the stiffness and responsiveness of the system. Recall the feeling of the stiffness increase in the brake pedal after you bleed the brakes (remove air bubbles).

If the air pressure is 20,000 kPa, then the volume ratio (at that pressure) is

$$x = \frac{20{,}000\text{ kPa}}{1.4 \times 10^6\text{ kPa}} = 0.0143 = 1.43\%$$

5.11 Resistance of capillary tube

The resistance is $$R = \frac{128\,\mu\,\ell}{\pi\,d^4}$$

for air $$R = \frac{128 \times 0.018 \times 10^{-6}\ \text{kPa s}\ 0.1\ \text{m}}{\pi\,0.5^4 \times 10^{-12}\ \text{m}^4} = 1.173 \times 10^6\ \frac{\text{kN s}}{\text{m}^5}$$

The maximum pressure drop is found from the maximum Reynolds number.

$$N_{r\,\max} = \frac{500{,}000}{\ell/d} = \frac{500{,}000}{100/0.5} = 2500$$

$$N_r = \frac{\bar{v}\,d}{\nu} = \frac{\frac{Q}{A}\,d}{\nu} = \frac{\frac{\delta P}{R\,A}\,d}{\nu}$$

The kinematic viscosity for air is $\nu = \mu/\rho$ is $15 \times 10^{-6}\ \frac{\text{m}^2}{\text{s}}$

Thus, the maximum pressure drop for air is

$$\delta P = \frac{R\,A\,\nu}{d}\,N_{r\,\max} = 1.173 \times 10^6\ \frac{\text{kN s}}{\text{m}^5}\,\frac{\pi}{4}\,0.0005\ \text{m}\ 15 \times 10^{-6}\ \frac{\text{m}^2}{\text{s}}\,2500$$

$$\delta P = 17.3\ \text{kPa} = 2.51\ \text{psi}$$

For oil, the resistance is $$R = \frac{128 \times 17 \times 10^{-6}\ \text{kPa s}\ 0.1\ \text{m}}{\pi\,0.5^4 \times 10^{-12}\ \text{m}^4} = 1108 \times 10^6\ \frac{\text{kN s}}{\text{m}^5}$$

The maximum pressure drop for oil is

$$\delta P = 1108 \times 10^6\ \frac{\text{kN s}}{\text{m}^5}\,\frac{\pi}{4}\,0.0005\ \text{m}\ 20 \times 10^{-6}\ \frac{\text{m}^2}{\text{s}}\,2500$$

$$\delta P = 21.8\ \text{MPa} = 3155\ \text{psi}$$

5.13 Water flow in an orifice

The flow through an orifice $Q = C_d\, A \sqrt{\dfrac{2\,\delta P}{\rho}}$ where $A = \dfrac{\pi}{4} d^2$

using numbers

$$Q = C_d \frac{\pi}{4} d^2 \sqrt{\frac{2\,\delta P}{\rho}} = 0.6 \frac{\pi}{4} 0.050^2\,\text{in}^2 \sqrt{\frac{2 \times 1000 \dfrac{\text{lbf}}{\text{in}^2}\, \dfrac{\text{lbm}\,32.18 \dfrac{\text{ft}}{\text{s}^2}}{\text{lbf}}\, 144 \dfrac{\text{in}^2}{\text{ft}^2}}{62.4 \dfrac{\text{lbm}}{\text{ft}^3}}}\, \frac{12\,\text{in}}{\text{ft}}$$

$$Q = 5.45 \frac{\text{in}^3}{\text{s}} = 1.415\,\text{gpm}$$

5.15 Critical pressure ratio

The term in the square root of the compressible flow equation is $(P_r)^{\frac{2}{k}} - (P_r)^{\frac{k+1}{k}}$

The maximum of this function occurs when the derivative is zero.

The derivative of this term is $\dfrac{\partial}{\partial P_r}\left[P_r^{\frac{2}{k}} - P_r^{\frac{k+1}{k}} \right] = 0$

or $\dfrac{2}{k} P_r^{\frac{2-k}{k}} - \dfrac{(k+1)}{k} P_r^{\frac{1}{k}} = 0$

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rearranging $\left(\dfrac{2}{k+1}\right)^k = \dfrac{P_r}{P_r^{2-k}} = P_r^{k-1}$

Thus the critical pressure ratio is $P_{cr} = \left(\dfrac{2}{k+1}\right)^{\frac{k}{k-1}}$

Solution for Two-Tank Problem

Assume that $h_1 > h_2$ so that the mass flow rate q_{m1} is positive if flowing from tank 1 to tank 2. Conservation of mass applied to each tank gives

$$\rho A_1 \dot{h}_1 = \rho q_v - q_{m1}$$

$$q_{m1} = \frac{\rho g}{R_1}(h_1 - h_2)$$

$$\rho A_2 \dot{h}_2 = q_{m1} - q_{mo}$$

$$q_{mo} = \frac{\rho g}{R_2} h_2$$

Substituting for q_{m1} and q_{mo}, and dividing by ρ gives the desired model.

$$A_1 \dot{h}_1 = q_v - \frac{g}{R_1}(h_1 - h_2)$$

$$A_2 \dot{h}_2 = \frac{g}{R_1}(h_1 - h_2) - \frac{g}{R_2} h_2$$

The density ρ does not appear explicitly in the final model because of the incompressibility assumption. However, the values of the resistances R_1 and R_2 depend on the density ρ.