HW 4: 4.9, 4.13, 4.16, 4.17, 4.25, 4.26, 4.30 and Case Study Simulation

4.9 Equivalent resistance

$$R_{eq} = \frac{R_5\left[\frac{R_1 R_2}{(R_1+R_2)} + \frac{R_3 R_4}{(R_3+R_4)}\right]}{R_5 + \frac{R_1 R_2}{(R_1+R_2)} + \frac{R_3 R_4}{(R_3+R_4)}}$$

4.13 *RLC* circuit analysis

Component equations

$$i_{R_1} = \frac{e_0 - e_1}{R_1}$$

$$i_{R_2} = \frac{e_1}{R_2}$$

$$i_L = \frac{e_1 - e_2}{L\,D} \qquad \text{with} \quad i_L(0)$$

$$i_{C_2} = C_2\,De_2 \qquad \text{with} \quad e_2(0)$$

Node equations

$$i_{R_1} = i_{R_2} + i_L$$

$$i_L = i_{C_2}$$

Substitute component equations into node equations

$$\frac{e_0 - e_1}{R_1} = \frac{e_1}{R_2} + \frac{e_1 - e_2}{L\,D} \quad \Rightarrow \quad \left[\frac{L\,D}{R_2} + \frac{L\,D}{R_1} + 1\right] e_1 = \frac{L\,D}{R_1} e_0 + e_2$$

$$\frac{e_1 - e_2}{L\,D} = C_2\,De_2 \quad \Rightarrow \quad \left[L\,C_2\,D^2 + 1\right] e_2 = e_1$$

Combine the above two equations

$$\left[L\,D\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+1\right]\left[L\,C_2\,D^2+1\right]e_2 - e_2 \;=\; \frac{L\,D}{R_1}\,e_0$$

Reduce

$$\left[L\,C_2\,D^2+\frac{R_1\,R_2\,C_2}{(R_1+R_2)}\,D+1\right]e_2 \;=\; \left(\frac{R_2}{R_1+R_2}\right)e_0$$

The known initial conditions are $i_L(0)$ and $e_2(0)$; we need $e_2(0)$ and $\dot{e}_2(0)$.

$$e_2(0) \;=\; \text{known}$$

$$\dot{e}_2(0) \;=\; \frac{1}{C_2}\,i_{C_2}(0) \;=\; \frac{1}{C_2}\,i_L(0) \quad \text{(from component equation and node equation)}$$

natural frequency $\qquad \omega_n \;=\; \dfrac{1}{\sqrt{L\,C_2}}$

damping ratio $\qquad \dfrac{2\,\zeta}{\omega_n} = \dfrac{R_1\,R_2\,C_2}{(R_1+R_2)} \qquad \Rightarrow \qquad \zeta = \dfrac{R_1\,R_2}{2\,(R_1+R_2)\sqrt{\dfrac{L}{C_2}}}$

static gain $\qquad G_s \;=\; \dfrac{R_2}{R_1+R_2}$

4.16 *RLC* circuit transfer function ____

Component equations

$$i_{R_1} = \frac{e_0 - e_1}{R_1}$$

$$i_L = \frac{e_1 - e_2}{L\,D} \quad \text{with } i_L(0)$$

$$i_{C_1} = C_1\,De_2 \quad \text{with } e_2(0)$$

$$i_{R_2} = \frac{e_2 - e_3}{R_2}$$

$$i_{C_2} = C_2\,De_3 \quad \text{with } e_3(0)$$

Node equations

$$i_{R_1} = i_L\,, \qquad i_L = i_{C_1} + i_{R_2}\,, \qquad i_{R_2} = i_{C_2}$$

Substitute component equations into node equations.

$$\frac{e_0 - e_1}{R_1} = \frac{e_1 - e_2}{LD} \qquad \text{or} \qquad \left[\frac{L}{R_1}D + 1\right]e_1 = \frac{LD}{R_1}e_0 + e_2$$

$$\frac{e_1 - e_2}{L\,D} = C_1\,De_2 + \frac{e_2 - e_3}{R_2} \qquad \text{or} \qquad \left[L\,C_1\,D^2 + \frac{L}{R_2}D + 1\right]e_2 = \frac{LD}{R_2}e_3 + e_1$$

$$\frac{e_2 - e_3}{R_2} = C_2\,De_3 \qquad \text{or} \qquad \left[R_2C_2\,D + 1\right]e_3 = e_2$$

Reduce to get e_3 as a function of e_0.

$$\frac{e_3}{e_0} = \frac{1}{R_2C_2LC_1D^3 + \left(LC_1 + LC_2 + R_1C_1R_2C_2\right)D^2 + \left(R_1C_1 + R_2C_2 + R_1C_2\right)D + 1}$$

4.17 *RLC* circuit state-space derivation __________

Component equations

$$i_{R_1} = \frac{e_0 - e_1}{R_1}$$

$$i_L = \frac{e_1 - e_2}{L\,D} \quad \text{with } i_L(0)$$

$$i_{C_1} = C_1\,De_2 \quad \text{with } e_2(0)$$

$$i_{R_2} = \frac{e_2 - e_3}{R_2}$$

$$i_{C_2} = C_2\,De_3 \quad \text{with } e_3(0)$$

Node equations

$$i_{R_1} = i_L\,, \qquad i_L = i_{C_1} + i_{R_2}\,, \qquad i_{R_2} = i_{C_2}$$

From the component equations, we can select state variables as follows.

$$u_1 = e_0$$

$$x_1 = i_L \qquad \text{thus} \quad \dot{x}_1 = Di_L = \frac{e_1 - e_2}{L} = \frac{1}{L}e_1 - \frac{1}{L}x_2$$

$$x_2 = e_2 \qquad \text{thus} \quad \dot{x}_2 = De_2 = \frac{1}{C_1}i_{C_1}$$

$$x_3 = e_3 \qquad \text{thus} \quad \dot{x}_3 = De_3 = \frac{1}{C_2}i_{C_2}$$

We need to eliminate e_1, i_{C1}, and i_{C2}. From the R_1 component equation and the first node equation,

$$e_1 = e_0 - R_1\,i_{R_1} = e_0 - R_1\,i_L = u_1 - R_1\,x_1$$

From the second node equation and the R_2 component equation,

$$i_{C_1} = i_L - i_{R_2} = i_L - \frac{e_2 - e_3}{R_2} = x_1 - \frac{1}{R_2}x_2 + \frac{1}{R_2}x_3$$

From the last node equation and the R_2 component equation,

$$i_{C_2} = i_{R_2} = \frac{e_2 - e_3}{R_2} = \frac{1}{R_2}x_2 - \frac{1}{R_2}x_3$$

State-space representation

$$\dot{x}_1 = -\frac{R_1}{L}x_1 - \frac{1}{L}x_2 + \frac{1}{L}u_1$$

$$\dot{x}_2 = \frac{1}{C}x_1 - \frac{1}{R_2C_1}x_2 + \frac{1}{R_2C_1}x_3$$

$$\dot{x}_3 = \frac{1}{R_2C_2}x_2 - \frac{1}{R_2C_2}x_3$$

With the initial conditions

$$x_1(0) = i_L(0)$$

$$x_2(0) = e_2(0)$$

$$x_3(0) = e_3(0)$$

4.25 Dual op-amp filter circuit

The two op-amp circuits can be treated independently since the output of each acts like a voltage source to the next stage. For the first op-amp, the two resistances R are the same thus, the circuit is an inverter.

$$e_1 = -e_i$$

The second op-amp circuit is a low-pass filter.

$$e_2 = \frac{-\dfrac{R_f}{R_i}}{R_f\, C_f\, D + 1}\, e_1$$

The second op-amp drives the RC circuit.

$$e_o = \frac{1}{R_L\, C_L\, D + 1}\, e_2$$

Combining,

$$e_o = \left[\frac{1}{R_L\, C_L\, D+1}\right]\left[\frac{\dfrac{R_f}{R_i}}{R_f\, C_f\, D+1}\right] e_i = \frac{\dfrac{R_f}{R_i}\, e_i}{R_L\, C_L\, R_f\, C_f\, D^2 + \left(R_L\, C_L + R_f\, C_f\right) D + 1}$$

The static gain is $\qquad G_s = \dfrac{R_f}{R_i}$

Using $\tau_L = R_L\, C_L$ and $\tau_f = R_f\, C_f$

The natural frequency is $\omega_n = \sqrt{\dfrac{1}{R_L\, C_L\, R_f\, C_f}} = \sqrt{\dfrac{1}{\tau_L\, \tau_f}}$

The damping ratio is $\zeta = \dfrac{\left(R_L\, C_L + R_f\, C_f\right)}{2\sqrt{R_L\, C_L\, R_f\, C_f}} = \dfrac{\left(\tau_L + \tau_f\right)}{2\sqrt{\tau_L\, \tau_f}}$

Using the values stated, $\dfrac{R_f}{R_i} = 1$

$$\tau_L = 500 \text{ ohm } 10\times10^{-6} \frac{\text{s}}{\text{ohm}} = 0.005 \text{ s}$$

$$\tau_f = 10\times10^{3} \text{ ohm } 1\times10^{-6} \frac{\text{s}}{\text{ohm}} = 0.010 \text{ s}$$

Thus, the static gain is $G_s = 1$

The natural frequency is $\omega_n = \sqrt{\dfrac{1}{0.005 \text{ s}\times 0.010 \text{ s}}} = 27.1 \dfrac{\text{rad}}{\text{s}} = 4.32 \text{ Hz}$

The damping ratio is $\zeta = \dfrac{(0.005+0.010)}{2\sqrt{0.005\times0.010}} = 0.204$

4.26 Dual op-amp filter state-space derivation ______________

From the analysis in Problem 4.25, the modeling equations are

$$e_2 = \frac{G_s}{\tau_f\, D+1}\, e_i \qquad \text{with} \quad e_2(0)$$

$$e_o = \frac{1}{\tau_L\, D+1}\, e_2 \qquad \text{with} \quad e_o(0)$$

where $G_s = \dfrac{R_f}{R_i}$ $\quad \tau_L = R_L\, C_L$ and $\quad \tau_f = R_f\, C_f$

The state-space representation for this system is

$$u_1 = e_i$$

$$x_1 = e_2 \qquad \text{thus} \qquad \dot{x}_1 = De_2 = \frac{-x_1 + G_s u_1}{\tau_f}$$

$$x_2 = e_o \qquad \text{thus} \qquad \dot{x}_2 = De_o = \frac{-x_2 + x_1}{\tau_L}$$

4.30 Voltmeter measurement degradation ________________________________

The transfer function of the original circuit is

$$e_1 = \frac{1}{1 + \frac{10,000}{47,000}} e_0 = 0.8246\, e_0$$

The transfer function of the circuit with the voltmeter is

$$e_1 = \frac{1}{1 + \frac{10,000}{47,000}\left(1 + \frac{47,000}{R_{meter}}\right)} e_0$$

If $R_{meter} = 100$ kΩ, then voltage reading will be degraded by 7.6 % (i.e., volt reading = 92.4 % of undisturbed voltage).

If $R_{meter} = 1$ MΩ, then voltage reading will be degraded by 0.8 %.

Case Simulation Study: