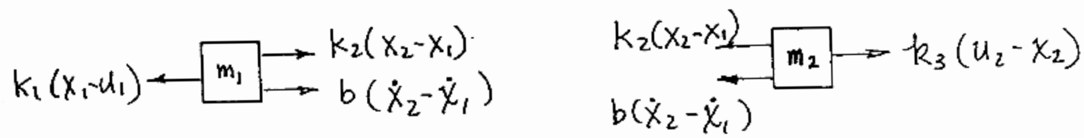
HW 3 Solution: 3.14, 3.21, 3.24,3.25, 3.31, 3.33, 3.36 and Case Study Simulation

3.14 Two degree of freedom mechanical system



a. Newton's second law applied to free bodies of the two masses gives:

$$k_2(x_2 - x_1) + b(\dot{x}_2 - \dot{x}_1) - k_1\,(x_1 - u_1) \;=\; m_1\,\ddot{x}_1$$

$$k_3(u_2 - x_2) - k_2(x_2 - x_1) \;-\; b(\dot{x}_2 - \dot{x}_1) = m_2\,\ddot{x}_2$$

Or

$$m_1\,\ddot{x}_1 + b\,\dot{x}_1 + k_1x_1 + \; k_2x_1 - b\,\dot{x}_2 - k_2x_2 = k_1u_1$$

$$m_2\,\ddot{x}_2 + b\,\dot{x}_2 + k_2x_2 + \; k_3x_2 - b\,\dot{x}_1 - k_2x_1 = k_3u_2$$

b. Employing the *D*-operator:

$$[m_1D^2 + bD + (k_1 + k_2)]\,x_1 - [bD + k_2]x_2 = k_1u_1$$

$$-[bD + k_2]x_1 + [m_2D^2 + bD + (k_2 + k_3)]\,x_2 \;= k_3u_2$$

c. Use Cramer's rule to solve for x_2

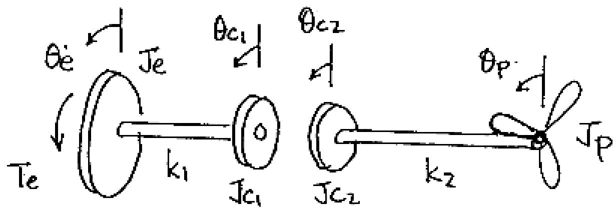
$$x_2 \;=\; \frac{[m_1D^2 + bD + (k_1 + k_2)]k_3u_2 + [bD + k_2]k_1u_1}{[m_1D^2 + bD + (k_1 + k_2)][m_2D^2 + bD + (k_2 + k_3)] - [bD + k_2]^2}$$

Since there are two inputs, the **static gains** are:

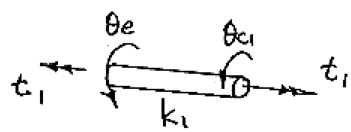
$$\frac{x_2}{u_1} = \frac{k_1k_2}{(k_1 + k_2)(k_2 + k_3) - k_2^{\,2}}$$

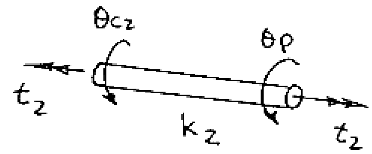
$$\frac{x_2}{u_1} = \frac{k_1k_2}{(k_1 + k_2)(k_2 + k_3) - k_2^{\,2}}$$

3.21 Boat engine & propeller



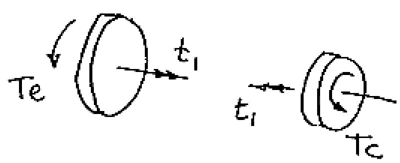
The torques carried by the two shafts are:





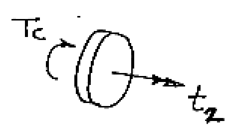
$$t_1 = k_1(\theta_{c1} - \theta_e) \qquad \text{and} \qquad t_2 = k_2(\theta_p - \theta_{c2})$$

The equations of motion for the four rotating masses are:

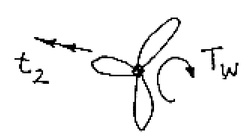


Engine: $$J_e\ddot{\theta}_e + k_1\theta_e - k_1\theta_{c1} = T_e(t)$$

Clutch plate 1: $$J_{c1}\ddot{\theta}_{c1} + k_1\theta_{c1} - k_1\theta_e = T_c$$



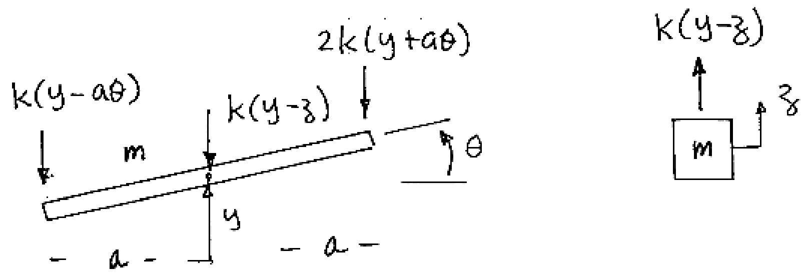
Clutch plate 2: $$J_{c2}\ddot{\theta}_{c2} + k_2\theta_{c2} - k_2\theta_p = -T_c$$



Propeller: $$J_p\ddot{\theta}_p + k_2\theta_p - k_2\theta_{c2} = -T_w$$

Here T_c is torque transmitted by the clutch and can be represented by $-b(\dot{\theta}_{c1} - \dot{\theta}_{c2})$. If the engine shaft is considered to be rigid, the first two equations can be combined to eliminate one angular coordinate.

3.24 Bar-mass system



Equations of motion of the bar are:

$\Sigma F_y = m\ddot{y}$ $\qquad -k(y-z) - k(y-a\theta) - 2k(y+a\theta) = m\ddot{y}$

$$m\ddot{y} + ky - kz + ky - ka\theta + 2ky + 2ka\theta = 0$$

$$m\ddot{y} + 4ky + ka\theta - kz = 0$$

$\Sigma M_{c.g.} = J_{c.g.}\ddot{\theta}$ $\qquad -2k(y+a\theta) + k(y-a\theta)a = J_{c.g.}\ddot{\theta}$

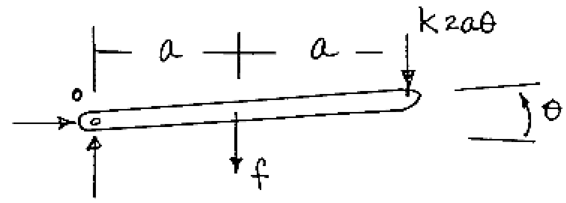
$$J_{c.g.}\ddot{\theta} + 3ka^2\theta + kay = 0$$

For the attached mass:

$\Sigma F_z = m\ddot{z}$ $\qquad m\ddot{z} + kz - ky = 0$

These equations may now be put into classical or state-space form.

3.25 Lever device



If f is the damper force, the free body of the lever gives:

$$\Sigma M_0 = J_0 \ddot{\theta} \qquad -2a\,k(2a)\theta - f a\,\text{sign}(\dot{\theta}) = J_0 \ddot{\theta}$$

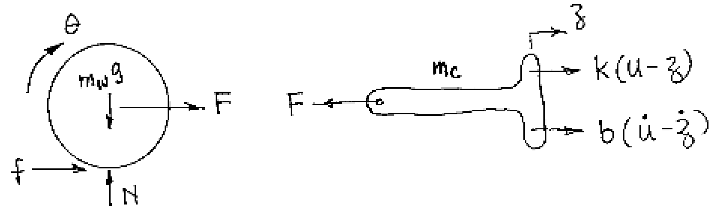
Classical form: $$J_0 \ddot{\theta} + 4ka^2\theta + f a\text{sign}(\dot{\theta}) = 0$$

State-space form: $$x_1 = \dot{\theta}, \qquad x_2 = \theta$$

$$\dot{x}_1 = \frac{[-4a^2 k x_2 - f a\,\text{sign}(x_1)]}{J_0}$$

$$\dot{x}_2 = x_1$$

3.31 Wheel with position input



a. For the wheel:

$$\Sigma M_{c.g.} = J_{c.g.}\ddot{\theta} \qquad -fr = J_{c.g.}\ddot{\theta}$$

$$\Sigma F_z = m_w \ddot{z} \qquad f + F = m_w \ddot{z}$$

For no slip $\quad z = r\theta \quad$ thus $\quad f + F = m_w r\ddot{\theta}$

$$f = m_w r\ddot{\theta} - F$$

Substitute into the angular equation of motion

$$-m_w r^2 \ddot{\theta} + Fr = J_{c.g.}\ddot{\theta} \qquad \text{and} \qquad F = \frac{J_{c.g}\ddot{\theta} + m_w r^2 \ddot{\theta}}{r}$$

For a uniform disk: $J_{c.g.} = \dfrac{1}{2} m_w r^2$ giving $\quad F = \dfrac{1.5 m_w \ddot{\theta} r^2}{r}$

For the mass m_c: $\quad \Sigma F_z = m_c \ddot{z} \qquad -F + k(u - z) + b(\dot{u} - \dot{z}) = m_c \ddot{z}$

But $\quad F = \dfrac{1.5 m_w \ddot{\theta} r^2}{r} \quad$ and $\quad \ddot{\theta} = \dfrac{\ddot{z}}{r} \quad$ thus $\quad F = 1.5 m_w \ddot{z}$

Substitution gives:

$$m_c \ddot{z} + 1.5 m_w \ddot{z} + b\dot{z} + kz = b\dot{u} + ku$$

b. Transfer function

$$\frac{z}{u} = \frac{bD + k}{(m_c + 1.5 m_w)D^2 + bD + k}$$

c. Static gain

$$\left(\frac{z}{u}\right)_{static} = \frac{k}{k} = 1$$

d. Slip

Slip occurs when $z \neq r\theta$. In this case z and θ must be treated as separate variables.

3.33 Bar-mass system by the Lagrange method ______________________

Refer to problem 3.24 solution

$$T = \frac{1}{2} m \dot{y}^2 + \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} m \dot{z}^2$$

And

$$U = \frac{1}{2} k (y - z)^2 + \frac{1}{2} k (y - a\theta)^2 + \frac{1}{2} 2k (y + a\theta)^2$$

$$\frac{\partial T}{\partial \dot{y}} = m \dot{y} \qquad \frac{\partial T}{\partial \dot{\theta}} = J \dot{\theta} \qquad \frac{\partial T}{\partial \dot{z}} = m \dot{z}$$

$$\frac{\partial U}{\partial y} = k(y - z) + k(y - a\theta) + 2k(y + a\theta)$$

$$\frac{\partial U}{\partial \theta} = -ak(y - a\theta) + 2ak(y + a\theta)$$

$$\frac{\partial U}{\partial z} = -k(y - z)$$

Combining these results:

$$m\ddot{y} + ky - kz + ky - ka\theta + 2ky + 2ka\theta = 0$$

Equation for y

$$m\ddot{y} + 4ky - kz + ka\theta = 0$$

$$J\ddot{\theta} - aky + ka^2\theta + 2aky + 2ka^2\theta = 0$$

Equation for θ

$$J\ddot{\theta} - kay + 3ka^2\theta = 0$$

Equation for z

$$m\ddot{z} + kz - ky = 0$$

3.36 Wheel with position input by the Lagrange method __________

Refer to the solution of problem 3.31. The kinetic energy is given by:

$$T = \frac{1}{2} m_w \dot{z}^2 + \frac{1}{2} J_{cg} \dot{\theta}^2 + \frac{1}{2} m_c \dot{z}^2$$

The no slip condition gives: $z = r\theta \qquad \theta = \dfrac{z}{r}$

Or
$$T = \frac{1}{2} m_w \dot{z}^2 + \frac{1}{2} J_{cg} \left(\frac{\dot{z}}{r} \right)^2 + \frac{1}{2} m_c \dot{z}^2$$

The potential and dissipation functions are:

$$U = \frac{1}{2} k (u - z)^2 \qquad\qquad R = \frac{1}{2} b (\dot{u} - \dot{z})$$

Use $J_{cg} = \dfrac{1}{2} m_w r^2$ and calculate the required derivatives:

$$\frac{\partial T}{\partial \dot{z}} = m_w \dot{z} + J_{cg} \frac{1}{r^2} \dot{z} + m_c \dot{z} \qquad\qquad \frac{\partial T}{\partial \dot{z}} = 1.5 m_w \dot{z} + m_c \dot{z}$$

$$\frac{\partial U}{\partial z} = -k (u - z) \qquad\qquad \frac{\partial R}{\partial \dot{z}} = -b (\dot{u} - \dot{z})$$

Combining these results gives:

$$(m_c + 1.5 m_w) \ddot{z} + b \dot{z} + kz = b\dot{u} + ku$$

Case study Simulation: we discussed the simulation results in class