Linearization and Review of Stability

ME584

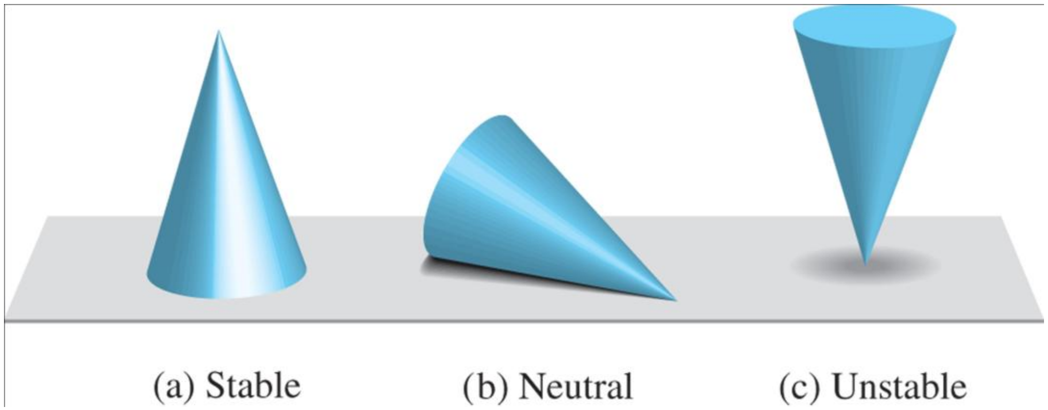
Fall 2010

Lecture Objectives and Activities

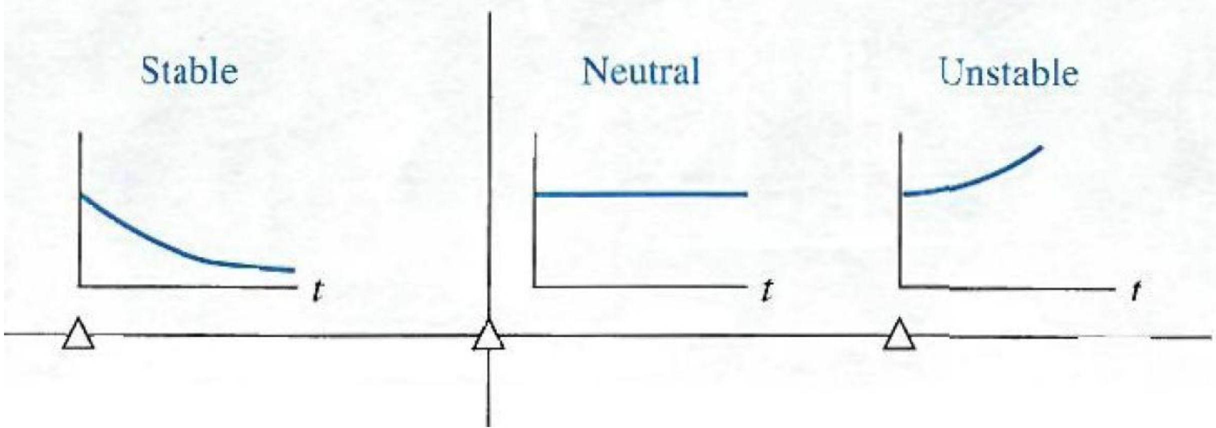
- Review of stability
- Importance of linearization
- Linearization of nonlinear systems
- Active learning activities
  - Pair-share problems

Definition of Stability

A stable system is a system with a bounded response to a bounded input



Response to a displacement/initial condition will produce either a decreasing, neural, or increasing response.



Stability Analysis – 1st Order ODE

$$1^{st} - Order : a_1 \frac{dx}{dt} + a_0 x = b_0 u$$

$$Characteristic\ equation:$$

$$a_1\lambda + a_0 = 0 \Rightarrow \lambda = -a_0 / a_1$$

$$System\ is\ stable\ if\ \lambda < 0,\ unstable\ if\ \lambda > 0$$

$$Example:\ 6\dot{x} + 2x = 2u;\ u = 0, x_o = 1$$

$$Characteristic\ equation: 6\lambda + 2 = 0 \Rightarrow \lambda = -3$$

$$x = x_o e^{-t/3}$$

$$Example:\ 6\dot{x} - 2x = 2u;\ u = 0, x_o = 1$$

$$Characteristic\ equation: 6\lambda - 2 = 0 \Rightarrow \lambda = 3$$

$$x = x_o e^{t/3}$$

Stability Analysis – 2nd Order ODE

$$2^{nd} - Order: a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = b_0 u$$

Characteristic Equation:

$$Let\ \frac{1}{\omega_n^2} = \frac{a_2}{a_o}, \frac{2\zeta}{\omega_n} = \frac{a_1}{a_o},$$

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

$$\lambda_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

System is unstable if λ_1 *and/or* $\lambda_2 > 0$

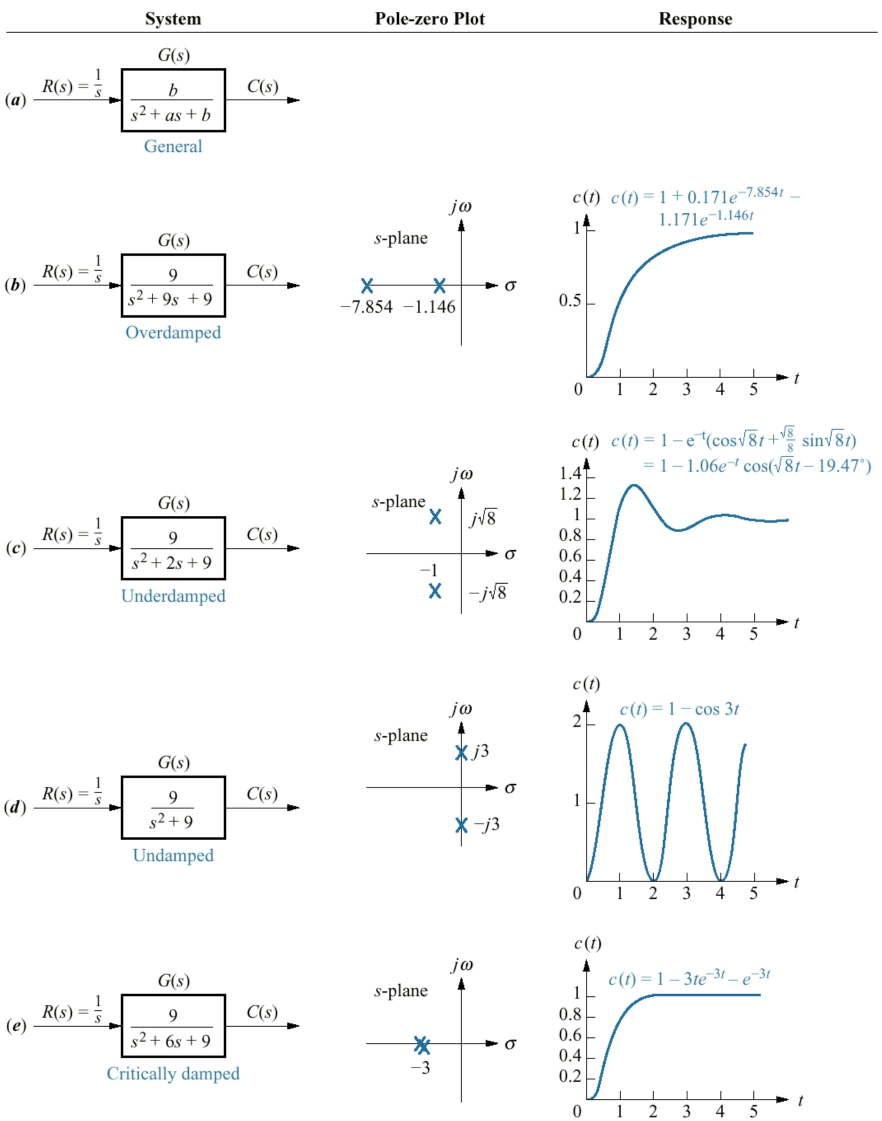
Stable response

$\zeta < 1$: *Underdamped (Oscilliation)*

$\zeta > 1$: *Overdamped (No oscilliation)*

$\zeta = 1$: *Critically damped (No oscilliation)*

Relative stability: degree of stability



Stability Analysis – State Space (SS)

State space format

$$\dot{x} = Ax$$

Let $x = ke^{\lambda t}$, *substitute*

$$\lambda k e^{\lambda t} = A k e^{\lambda t} \quad or \quad \lambda x = Ax$$

$$(\lambda I - A)x = 0$$

Non-trivial solution if

$$\det(\lambda I - A) = 0$$

Stability Analysis with SS - Example

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -\alpha & -\beta & 0 \\ \beta & -\gamma & 0 \\ \alpha & \gamma & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The characteristic equation is then

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \det\left\{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -\alpha & -\beta & 0 \\ \beta & -\gamma & 0 \\ \alpha & \gamma & 0 \end{bmatrix}\right\}$$

$$= \det\begin{bmatrix} \lambda + \alpha & \beta & 0 \\ -\beta & \lambda + \gamma & 0 \\ -\alpha & -\gamma & \lambda \end{bmatrix}$$

$$= \lambda[(\lambda + \alpha)(\lambda + \gamma) + \beta^2]$$

$$= \lambda[\lambda^2 + (\alpha + \gamma)\lambda + (\alpha\gamma + \beta^2)] = 0$$

system is stable when $\alpha + \gamma > 0$ and $\alpha\gamma + \beta^2 > 0$.

Importance of linearization

- Dynamics Analysis
- Control and estimation systems design

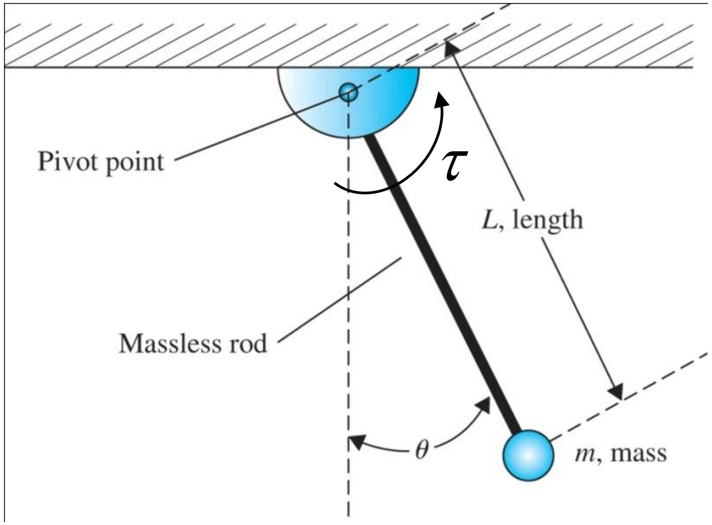
Importance of linearization

- Is nature linear or nonlinear? physical systems?
  - Many physical systems behave linearly within some range of variable, but become nonlinear as variables increase without limit
  - Possible to linearize nonlinear systems
- Example: Pendulum

$$m\ddot{\theta} + K\dot{\theta} + \frac{mg\sin(\theta)}{L} = \tau$$

$$For\ small\ \theta,\ \sin(\theta) = \theta$$

$$m\ddot{\theta} + K\dot{\theta} + (mg/L)\theta = \tau$$



- Tractable analysis with linear model

Importance of linearization

- Is this system stable? $m\ddot{\theta} + K\dot{\theta} + (mg/l)\theta = \tau$
- State space model:

$$state\ variables\ \ x = [\theta\ \ \dot{\theta}]$$

$$\dot{x} = Ax + Bu$$

$$where\ \ A = \begin{bmatrix} 0 & 1 \\ -g/l & -K/m \end{bmatrix};\ \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ u = \tau$$

- Control

$$u = -Gx$$

$$Where\ G\ is\ the\ control\ matrix,$$

$$\dot{x} = Ax + Bu = Ax + B(-Gx) = (A - BG)x$$

$$Choose\ G\ to\ achieve\ desired\ performance$$

- *Estimation*

$$u = -G\hat{x}$$

$$where\ \hat{x}\ is\ estimate\ of\ x$$

Linear Approximation

$$\dot{x} = \frac{dx}{dt} = f(x,u)$$

$$x = \bar{x} + x^*$$

$$u = \bar{u} + u^*$$

$\bar{x}$: *equilibrium value of x about which linearization is taken*

also called (steady state value/nominal value)

$\bar{u}$: *equilibrium value of u about which linearization is taken*

x^* : *small perturbation or variation of x*

u^* : *small perturbation or variation of u*

To solve for $\bar{x}$ *and* $\bar{u}$,

set $f(\bar{x},\bar{u}) = 0$

$\bar{x}$ *and* $\bar{u}$ *can also be provided from testing*

Taylor's Expansion

$$x = \bar{x} + x^*$$

$$\frac{dx}{dt} = 0 + \frac{dx^*}{dt} = f(\bar{x}, \bar{u}) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} x^* + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} u^* + H.O.T.$$

$$\frac{dx^*}{dt} \cong Ax^* + Bu^*$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} \quad and \quad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}}$$

A and B are called Jacobian matrices

Linearization Procedure

Step 1.

$If\ \bar{x}\ and\ \bar{u}\ are\ not\ specified,\ set\ \dot{x}=0\ to\ solve\ for\ \bar{x}\ and\ \bar{u}$

$Define\ x_1, x_2, \ldots, x_n\ and\ u_1, u_2, \ldots, u_m,\ and\ form\ f_1, f_2, \ldots, f_n$

Step 2.

$Solve\ for\ A\ and\ B.\ \ Assume\ A\ is\ (n\times n)\ and\ B\ is\ (n\times m)$

$$A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} = \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{\bar{x},\bar{u}} & \left.\frac{\partial f_1}{\partial x_2}\right|_{\bar{x},\bar{u}} & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_{\bar{x},\bar{u}} \\ \left.\frac{\partial f_2}{\partial x_1}\right|_{\bar{x},\bar{u}} & \left.\frac{\partial f_2}{\partial x_2}\right|_{\bar{x},\bar{u}} & \cdots & \left.\frac{\partial f_2}{\partial x_n}\right|_{\bar{x},\bar{u}} \\ \vdots & \vdots & \vdots & \vdots \\ \left.\frac{\partial f_n}{\partial x_1}\right|_{\bar{x},\bar{u}} & \left.\frac{\partial f_n}{\partial x_2}\right|_{\bar{x},\bar{u}} & \cdots & \left.\frac{\partial f_n}{\partial x_n}\right|_{\bar{x},\bar{u}} \end{bmatrix} \quad and \quad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} = \begin{bmatrix} \left.\frac{\partial f_1}{\partial u_1}\right|_{\bar{x},\bar{u}} & \left.\frac{\partial f_1}{\partial u_2}\right|_{\bar{x},\bar{u}} & \cdots & \left.\frac{\partial f_1}{\partial u_m}\right|_{\bar{x},\bar{u}} \\ \left.\frac{\partial f_2}{\partial u_1}\right|_{\bar{x},\bar{u}} & \left.\frac{\partial f_2}{\partial u_2}\right|_{\bar{x},\bar{u}} & \cdots & \left.\frac{\partial f_2}{\partial u_m}\right|_{\bar{x},\bar{u}} \\ \vdots & \vdots & \vdots & \vdots \\ \left.\frac{\partial f_n}{\partial u_1}\right|_{\bar{x},\bar{u}} & \left.\frac{\partial f_n}{\partial u_2}\right|_{\bar{x},\bar{u}} & \cdots & \left.\frac{\partial f_n}{\partial u_m}\right|_{\bar{x},\bar{u}} \end{bmatrix}$$

Step 3.

$$Form\ \frac{dx^*}{dt} \cong Ax^* + Bu^*$$

Example

$$m\ddot{\theta} + K\dot{\theta} + \frac{mg\sin(\theta)}{L} = 0$$

$$let\ x_1 = \theta,\ x_2 = \dot{\theta},\ u = 0\ (no\ control\ \ input)$$

$$f = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{Bmatrix} x_2 \\ -\frac{k}{m}x_2 - \frac{g}{L}\sin x_1 \end{Bmatrix}$$

Step 1

$$Set\ \dot{x} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = 0,$$

$$\dot{x}_1 = \bar{x}_2 = 0 \Rightarrow \bar{x}_2 = 0$$

$$\dot{x}_2 = \frac{-k}{m}\bar{x}_2 - \frac{g}{L}\sin\bar{x}_1 = 0$$

$$\Rightarrow \bar{x}_1 = 0, \pi, 2\pi$$

$$For\ this\ example, let\ us\ consider\ \bar{x}_1 = 0$$

$$\bar{x} = \begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Step 2

$$A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} = \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{\bar{x},\bar{u}} & \frac{\partial f_1}{\partial x_2}\, x_2 \\ \left.\frac{\partial f_2}{\partial x_1}\right|_{\bar{x},\bar{u}} & \left.\frac{\partial f_2}{\partial x_2}\right|_{\bar{x},\bar{u}} \end{bmatrix}$$

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{Bmatrix} x_2 \\ -\frac{k}{m}x_2 - \frac{g}{L}\sin x_1 \end{Bmatrix}$$

$$\left.\frac{\partial f_1}{\partial x_1}\right|_{\bar{x},\bar{u}} = \left.\frac{\partial}{\partial x_1}(x_2)\right|_{\bar{x},\bar{u}} = 0$$

$$\left.\frac{\partial f_1}{\partial x_2}\right|_{\bar{x},\bar{u}} = \left.\frac{\partial}{\partial x_2}(x_2)\right|_{\bar{x},\bar{u}} = 1$$

$$\left.\frac{\partial f_2}{\partial x_1}\right|_{\bar{x},\bar{u}} = \left.\frac{\partial}{\partial x_1}\left(-\frac{k}{m}x_2 - \frac{g}{L}\sin x_1\right)\right|_{\bar{x},\bar{u}} = -\frac{g}{L}\cos x_1\Big|_{\bar{x}_1=0} = -\frac{g}{L}$$

$$\left.\frac{\partial f_2}{\partial x_2}\right|_{\bar{x},\bar{u}} = \left.\frac{\partial}{\partial x_2}\left(-\frac{k}{m}x_2 - \frac{g}{L}\sin x_1\right)\right|_{\bar{x},\bar{u}} = -\frac{k}{m}$$

Step 2 (continued)

$$B = \left. \frac{\partial f}{\partial u} \right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = 0$$

Step 3

$$\frac{dx^*}{dt} = \begin{Bmatrix} \dot{x}_1^* \\ \dot{x}_2^* \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{k}{m} \end{bmatrix} \begin{Bmatrix} x_1^* \\ x_2^* \end{Bmatrix}$$

$$\dot{x}_1^* = x_2^*$$

$$\dot{x}_2^* = -\frac{g}{L} x_1^* - \frac{k}{m} x_2^*$$

$$\textit{The same as } \ddot{\theta} = -\frac{g}{L}\theta - \frac{k}{m}\dot{\theta}$$

Pair-Share Exercise

Linearize the system about the point where the mass compresses the spring by $1m$ *and the applied force* $u = 0$,

$$m\ddot{x} = u + mg - k_1 x - k_2 x^3$$

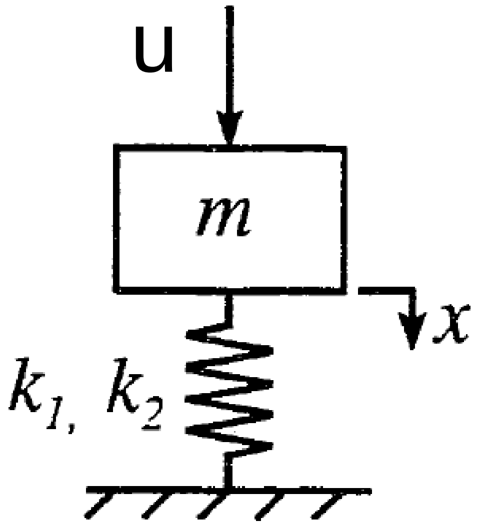
where,

$m = 200 kg$

$g = 10 m/s^2$

$k_1 = 1000 N/m$

$k_2 = 1000 N/m^3$



Step 1

$$Let\ x_1 = x\ and\ x_2 = \dot{x}$$

$$f = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{Bmatrix} x_2 \\ \dfrac{u}{m} + g - \dfrac{k_1}{m} x_1 - \dfrac{k_2}{m} x_1{}^3 \end{Bmatrix}$$

$$\bar{x} = \begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\bar{u} = 0$$

Step 2

$$A = \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial x_1}\right|_{\bar{x},\bar{u}} & \left.\dfrac{\partial f_1}{\partial x_2}\right|_{\bar{x},\bar{u}} \\ \left.\dfrac{\partial f_2}{\partial x_1}\right|_{\bar{x},\bar{u}} & \left.\dfrac{\partial f_2}{\partial x_2}\right|_{\bar{x},\bar{u}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k_1}{m} - \dfrac{3k_2}{m}\bar{x}_1^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -20 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial u}\right|_{\bar{x},\bar{u}} \\ \left.\dfrac{\partial f_2}{\partial u}\right|_{\bar{x},\bar{u}} \end{bmatrix} = \begin{bmatrix} 0 \\ .005 \end{bmatrix}$$

Step 3

$$\dot{x}^* = \begin{bmatrix} 0 & 1 \\ -20 & 0 \end{bmatrix} x^* + \begin{bmatrix} 0 \\ .005 \end{bmatrix} u^*$$

Pair-Share Example

A simple robot arm is modeled as

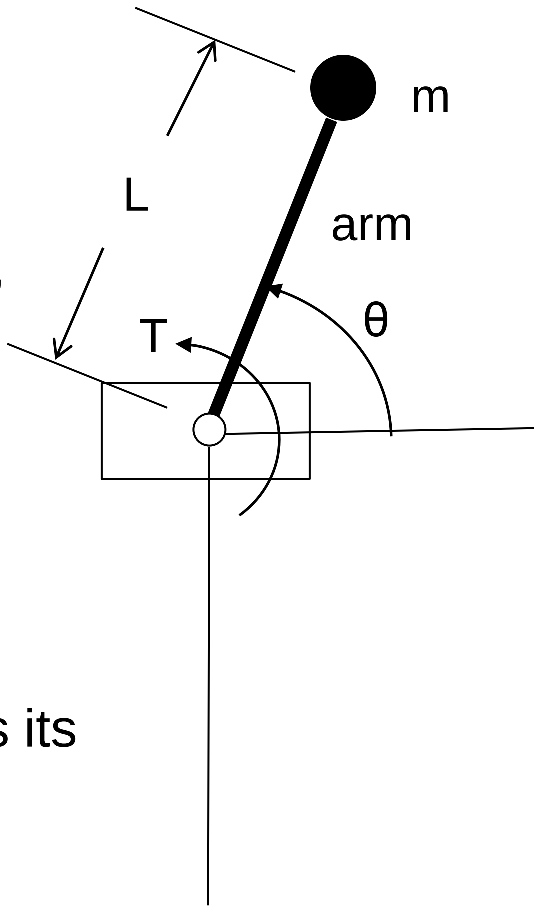
$$I\ddot{\theta} = T - mgL\cos\theta$$

where I is moment of inertia of arm, m is the mass, and T is the torque that the motor supplies.

We want the motor to hold the arm at five angles:

$$\theta_e = 0^o,\ 45^o,\ 90^o,\ 135^o,\ 180^o, 225^o$$

Find the torque required and determine what will happen if something hits the arm and slightly alters its position?



Step 1

$$Let\ \ x_1 = \theta,\ x_2 = \dot{\theta},\ u = T$$

$$f = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{Bmatrix} x_2 \\ \dfrac{u}{I} - \dfrac{mgL}{I}\cos x_1 \end{Bmatrix}$$

$$\bar{x} = \begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{Bmatrix} = \begin{Bmatrix} \theta_e \\ 0 \end{Bmatrix}$$

To find the torque $\bar{u}$ at $\bar{x}$, set $x = \bar{x}$ *and* $\dot{x}_1 = 0$ *and* $\dot{x}_2 = 0$

$$\bar{u} = mgL\cos x_1$$

$\bar{x}_1 = 0^o$,	$\bar{u} = mgL$
$\bar{x}_1 = 45^o$,	$\bar{u} = 0.707mgL$
$\bar{x}_1 = 90^o$,	$\bar{u} = 0$
$\bar{x}_1 = 135^o$,	$\bar{u} = -0.707mgL$
$\bar{x}_1 = 180^o$,	$\bar{u} = -mgL$
$\bar{x}_1 = 225^o$,	$\bar{u} = -0.707mgL$

Step 2

$$A = \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial x_1}\right|_{\bar{x},\bar{u}} & \left.\dfrac{\partial f_1}{\partial x_2}\right|_{\bar{x},\bar{u}} \\ \left.\dfrac{\partial f_2}{\partial x_1}\right|_{\bar{x},\bar{u}} & \left.\dfrac{\partial f_2}{\partial x_2}\right|_{\bar{x},\bar{u}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{mgL}{I}\sin \bar{x}_1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial u}\right|_{\bar{x},\bar{u}} \\ \left.\dfrac{\partial f_2}{\partial u}\right|_{\bar{x},\bar{u}} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{1}{I} \end{bmatrix}$$

Step 3

$$\dot{x}^* = \begin{Bmatrix} \dot{x}_1^* \\ \dot{x}_2^* \end{Bmatrix} = \left[\begin{bmatrix} 0 & 1 \\ \dfrac{mgL}{I}\sin \bar{x}_1 & 0 \end{bmatrix}\right] \begin{Bmatrix} x_1^* \\ x_2^* \end{Bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{I} \end{bmatrix} u^*$$

$$\dot{x}_2^* = \frac{mgL}{I}(\sin \bar{x}_1)\, x_1^* + \frac{u^*}{I}$$

When Arm is Hit

$$\dot{x}_2^* = \ddot{x}_1^*, so$$

$$\ddot{x}_1^* - \frac{mgL}{I}(\sin \bar{x}_1)\, x_1^* = \frac{u^*}{I}$$

Characteristic equation

$$\lambda^2 - \frac{mgL}{I}(\sin \bar{x}_1) = 0$$

has roots

$$\lambda_{1,2} = \pm\sqrt{\frac{mgL}{I}\sin \bar{x}_1}$$

For

$\bar{x}_1 = 0^o,\ \lambda_{1,2} = 0,0$ (*neutrally stable*)

$\bar{x}_1 = 45^o,\ \lambda_{1,2} = \pm\sqrt{\frac{.707mgL}{I}}$ (*unstable*)

$\bar{x}_1 = 90^o,\ \lambda_{1,2} = \pm\sqrt{\frac{mgL}{I}}$ (*unstable*)

$\bar{x}_1 = 135^o,\ \lambda_{1,2} = \pm\sqrt{\frac{.707mgL}{I}}$ (*unstable*)

$\bar{x}_1 = 180^o,\ \lambda_{1,2} = \pm\ 0,0$ (*neutrally stable*)

$\bar{x}_1 = 225^o,\ \lambda_{1,2} = \pm j\sqrt{\frac{.707mgL}{I}}$ (*undamped / neutrally stable*)

Lecture Recap

- Many nonlinear systems behave linearly with small perturbation
- Linearization procedure
  - Establish equilibrium
  - Solve for A and B
- Analysis is tractable with linear models
- Next lecture: Stability analysis and simulation with Matlab

References

- Woods, R. L., and Lawrence, K., Modeling and Simulation of Dynamic Systems, Prentice Hall, 1997.
- Palm, W. J., Modeling, Analysis, and Control of Dynamic Systems
- Close, C. M., Frederick, D. H., Newell, J. C., Modeling and Analysis of Dynamic Systems, Third Edition, Wiley, 2002