Exam 2

ME584 – Fall 2010

Instructions: The exam time is from 7:00PM to 8:15PM. This is a closed-book exam, and you will be allowed to use ONLY pencils/pens, erasers, and blank sheets to write your answers. NO calculator is allowed. Make sure you put page number on each answer sheet. All information needed to solve each problem is already included in that problem.

Relax!

Useful formula:

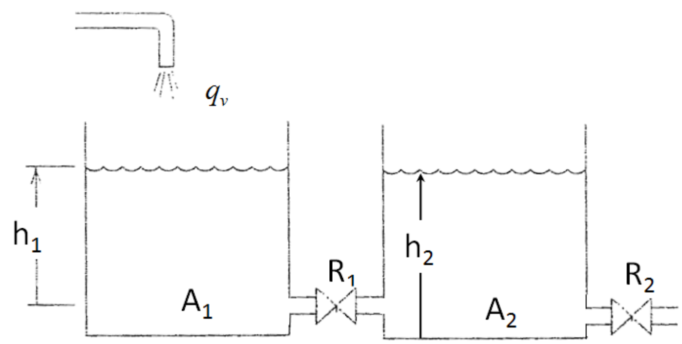
Problem 1 (25 points)

(a) Develop a model for the heights h_1 and h_2 in the liquid system shown in Figure below. The input volume flow rate q_v is given. Assume that laminar flow exists in the pipes. The laminar resistances are R_1 and R_2, and the bottom areas of the tanks are A_1 and A_2.

(b) Suppose the resistances are equal: $R=R_1=R_2$, and the areas are equal: $A=A_1=A_2$. Obtain the transfer function between h_2 and q_v, and answer the following questions:

   a. Suppose the inflow rate q_v is a unit-step function, find the steady state response for h_2, and estimate how long it will take to reach steady state.

   b. Is it possible to find a set of initial liquid heights so that the heights will oscillate up and down in the free response?



SOLUTION

(a) Assume that $h_1 > h_2$ so that the mass flow rate q_{m1} is positive if flowing from tank 1 to tank 2. Conservation of mass applied to each tank gives

$$\rho A_1 \dot{h}_1 = \rho q_v - q_{m1}$$

$$q_{m1} = \frac{\rho g}{R_1}(h_1 - h_2)$$

$$\rho A_2 \dot{h}_2 = q_{m1} - q_{mo}$$

$$q_{mo} = \frac{\rho g}{R_2} h_2$$

Substituting for q_{m1} and q_{mo}, and dividing by ρ gives the desired model.

$$A_1 \dot{h}_1 = q_v - \frac{g}{R_1}(h_1 - h_2) \tag{5.1-27}$$

(5 points)

$$A_2 \dot{h}_2 = \frac{g}{R_1}(h_1 - h_2) - \frac{g}{R_2} h_2 \tag{5.1-28}$$

The density ρ does not appear explicitly in the final model because of the incompressibility assumption. However, the values of the resistances R_1 and R_2 depend on the density ρ.

(b) Substituting $A_1 = A_2 = A$ and $R_1 = R_2 = R$ into the differential equations, we obtain

$$A \dot{h}_1 = q_v - \frac{g}{R}(h_1 - h_2)$$

$$A \dot{h}_2 = \frac{g}{R}(h_1 - h_2) - \frac{g}{R} h_2$$

Take the Laplace transform of each equation, assuming zero initial conditions, and collect terms.

$$\left(As + \frac{g}{R}\right)H_1(s) - \frac{g}{R}H_2(s) = Q_v(s) \tag{5.1-29}$$

$$-\frac{g}{R}H_1(s) + \left(As + \frac{2g}{R}\right)H_2(s) = 0 \tag{5.1-30}$$

Solve (5.1-30) for $H_1(s)$.

$$H_1(s) = \frac{R}{g}\left(As + \frac{2g}{R}\right)H_2(s)$$

Substitute this into (5.1-29).

$$\left(As + \frac{g}{R}\right)\left[\frac{R}{g}\left(As + \frac{2g}{R}\right)\right]H_2(s) - \frac{g}{R}H_2(s) = Q_v(s)$$

and solve for $H_2(s)/Q_v(s)$.

(5 points)

$$\frac{H_2(s)}{Q_v(s)} = \frac{gR}{R^2A^2s^2 + 3gRAs + g^2} \tag{5.1-31}$$

To answer the remaining questions, we must first determine if the system is stable. we can tell that the system is stable because the three coefficients of the denominator of the transfer function will always have the same sign because $A > 0$, $g > 0$, and $R > 0$. Thus the free response will decay to zero (that is, both tanks will become empty, which is no surprise!), and there will be a constant steady state response to a step input. This means that both heights will reach a constant value. Applying the final value theorem with $Q_v(s) = 1/s$, we obtain the steady state height.

(5 points)

$$h_{2ss} = \lim_{s\to 0} s\left[\frac{gR}{R^2A^2s^2 + 3gRAs + g^2}\right] = \frac{R}{g}$$

The characteristic equation is

$$R^2A^2s^2 + 3gRAs + g^2 = 0$$

and has the two real roots

$$s = \frac{g}{RA}\left(\frac{-3 \pm \sqrt{5}}{2}\right)$$

The time constants are

$$\tau_1 = \frac{RA}{g}\left(\frac{2}{3+\sqrt{5}}\right) = 0.382\frac{RA}{g}$$

and

$$\tau_2 = \frac{RA}{g}\left(\frac{2}{3-\sqrt{5}}\right) = 2.62\frac{RA}{g}$$

(5 points)

The dominant time constant is τ_2 and thus it will take a time equal to approximately $4\tau_2 = 10.5RA/g$ to reach steady state.

The free response cannot oscillate because both roots are real. Another way to determine this without computing the roots is to compute the damping ratio.

$$\zeta = \frac{3gRA}{2\sqrt{R^2A^2g^2}} = \frac{3}{2}$$

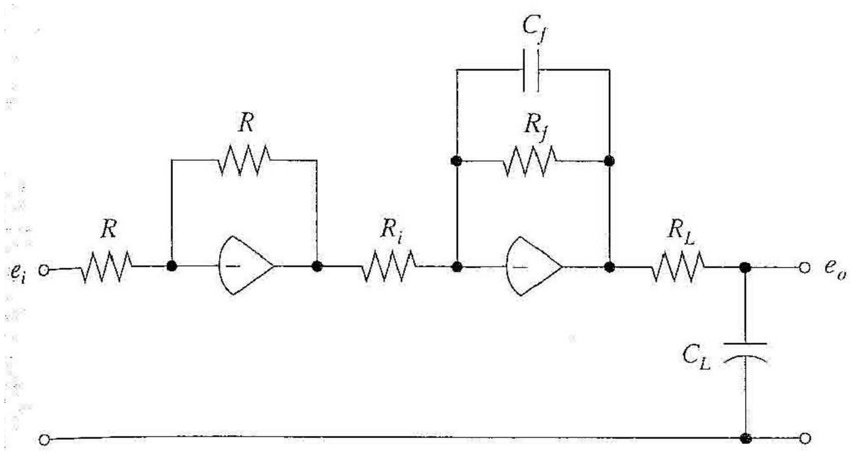
(5 points)

The free response will not oscillate because $\zeta > 1$.

Problem 2 (25 points):

Below is an electronic circuit with an op-amp buffer. Derive

a. The differential equation for e_0 as a function of the input e_i
b. The expression for the natural frequency and damping ratio
c. A state space representation of the differential equation



The two op-amp circuits can be treated independently since the output of each acts like a voltage source to the next stage. For the first op-amp, the two resistances R are the same thus, the circuit is an inverter.

$$e_1 = -e_i$$

The second op-amp circuit is a low-pass filter.

$$e_2 = \frac{-\dfrac{R_f}{R_i}}{R_f\, C_f\, D+1}\, e_1$$

The second op-amp drives the RC circuit.

$$e_o = \frac{1}{R_L\, C_L\, D+1}\, e_2$$

Combining,

$$e_o = \left[\frac{1}{R_L\, C_L\, D+1}\right]\left[\frac{\dfrac{R_f}{R_i}}{R_f\, C_f\, D+1}\right] e_i = \frac{\dfrac{R_f}{R_i}\, e_i}{R_L\, C_L\, R_f\, C_f\, D^2 + \left(R_L\, C_L + R_f\, C_f\right) D + 1}$$

(10 points)

The static gain is $\qquad G_s = \dfrac{R_f}{R_i}$

Using $\tau_L = R_L\, C_L$ and $\tau_f = R_f\, C_f$

The natural frequency is $\omega_n = \sqrt{\dfrac{1}{R_L\, C_L\, R_f\, C_f}} = \sqrt{\dfrac{1}{\tau_L\, \tau_f}}$ (5 points)

The damping ratio is $\zeta = \dfrac{(R_L\, C_L + R_f\, C_f)}{2\sqrt{R_L\, C_L\, R_f\, C_f}} = \dfrac{(\tau_L + \tau_f)}{2\sqrt{\tau_L\, \tau_f}}$ (5 points)

$$e_2 = \frac{G_s}{\tau_f\, D + 1} e_i \qquad \text{with} \quad e_2(0)$$

$$e_o = \frac{1}{\tau_L\, D + 1} e_2 \qquad \text{with} \quad e_o(0)$$

where $G_s = \dfrac{R_f}{R_i}$ $\quad \tau_L = R_L\, C_L$ and $\quad \tau_f = R_f\, C_f$

The state-space representation for this system is

$$u_1 = e_i$$

$$x_1 = e_2 \qquad \text{thus} \quad \dot{x}_1 = De_2 = \frac{-x_1 + G_s u_1}{\tau_f}$$

(5 points)

$$x_2 = e_o \qquad \text{thus} \quad \dot{x}_2 = De_o = \frac{-x_2 + x_1}{\tau_L}$$

Problem 3 (10 points)

Calculate the amount of heat loss through a window in a home. The temperature difference between the air inside and outside of the house is ΔT_{io}, with convection coefficient h_{out} and h_{in} on each side of the glass. The area of the glass is A, and the thickness is t.

The thermal resistance of the glass due to its conductivity is

$$R_k = \frac{t}{k\,A}$$

The total thermal resistance is composed of convection on the inside and outside of the glass, and the thermal conductivity resistance.

$$R_{Total} = \frac{1}{h_{in}\,A} + \frac{t}{k\,A} + \frac{1}{h_{out}\,A}$$

(9 points)

$$Q_h = \frac{\Delta T}{R_{Total}}$$

(1 point)

Problem 4 (10 points)

You are asked to develop a model for the temperature in the house that has five rooms. Without writing any equations, describe a model that can be used to calculate the temperature and the type of differential equations used for the modeling. Discuss the advantages/disadvantage of this model and all of your assumptions.

We can use the lumped parameter model approach to model the temperature in the house. This approach is based on the assumption that all the properties of thermal resistance and capacitance are lumped at selected points in space and produces a set of ordinary differential equation (ODE) in time. In this approach, we can model each room as a lump with the assumption that the temperature inside each room is uniform. This allows us to develop an ODE for each room, leading to several equations. The main advantage of this approach is that ODEs are much easier to analyze and solve analytically than partial differential equations, which are the actual modeling equations for these process. The disadvantage is that the solution that we obtain is an approximation.