Exam 1

ME584 – Fall 2010

Instructions: The exam time is from 7:00PM to 8:15PM. This is a closed-book exam, and you will be allowed to use ONLY pencils/pens, erasers, and blank sheets to write your answers. NO calculator is allowed. Make sure you put page number on each answer sheet. All information needed to solve each problem is already included in that problem.

Useful formula:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial U}{\partial q_i} = Q_i$$

where:

q_i : independent coordinates necessary to describe system's motion at any instant

Q_i : corresponding loading in each coordinate

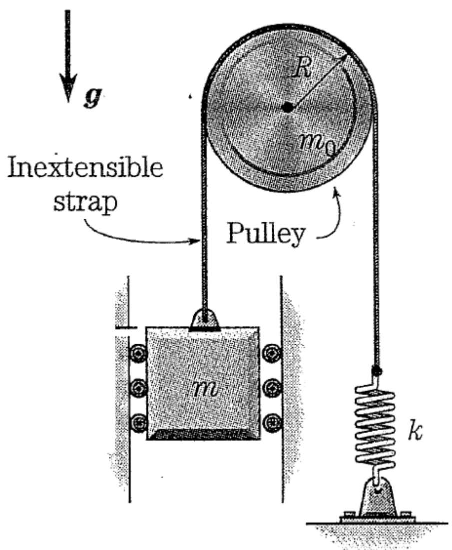
$U = f_1(q_i)$: potential energy in terms of coordinates

$T = f_2(\dot{q}_i^2)$: kinetic energy in terms of system masses, mass inertias, linear/angular velocities

$R = f_3(\dot{q}_i^2)$: energy dissipation due to viscous friction

Problem 1 (5 points)

Figure below shows a pulley of mass m_0 that is modeled as a disk, supported by an axle (which itself has bearings but is not shown) through its center. An ideal (that is, massless) inextensible strap over the disk supports a mass m at one of its ends and at the other end is restrained by an elastic element that is characterized by a linear spring of spring constant k. The disk does not slip relative to the strap. What is the order of this system?



There are two masses and one spring, however, because the disk moves without slipping, the distance that it rotates is equal to rθ, which is equal to the distance x traveled by the mass m. Thus, m and m_0 can be combined. Together with the spring, the system has two energy storage elements, and is a second order system.

Problem 2 (10 points): Express the following equations of motion in state space format.

$$2m\ddot{x} + mL\ddot{\theta} + 2kx = 0$$
$$mL\ddot{x} + mL^2\ddot{\theta} + mgl\theta = 0$$

$$2m\ddot{x} + mL\ddot{\theta} + 2kx = 0 \qquad (1)$$

$$mL\ddot{x} + mL^2\ddot{\theta} + mgl\theta = 0 \qquad (2)$$

First, we need to solve for $\ddot{x}$ *and* $\ddot{\theta}$. *Substituting* (1) *into* (2) *and solving for* $\ddot{\theta}$ *give*

$$\ddot{\theta} = \frac{2kx}{ml} - \frac{2g\theta}{l} \qquad (3)$$

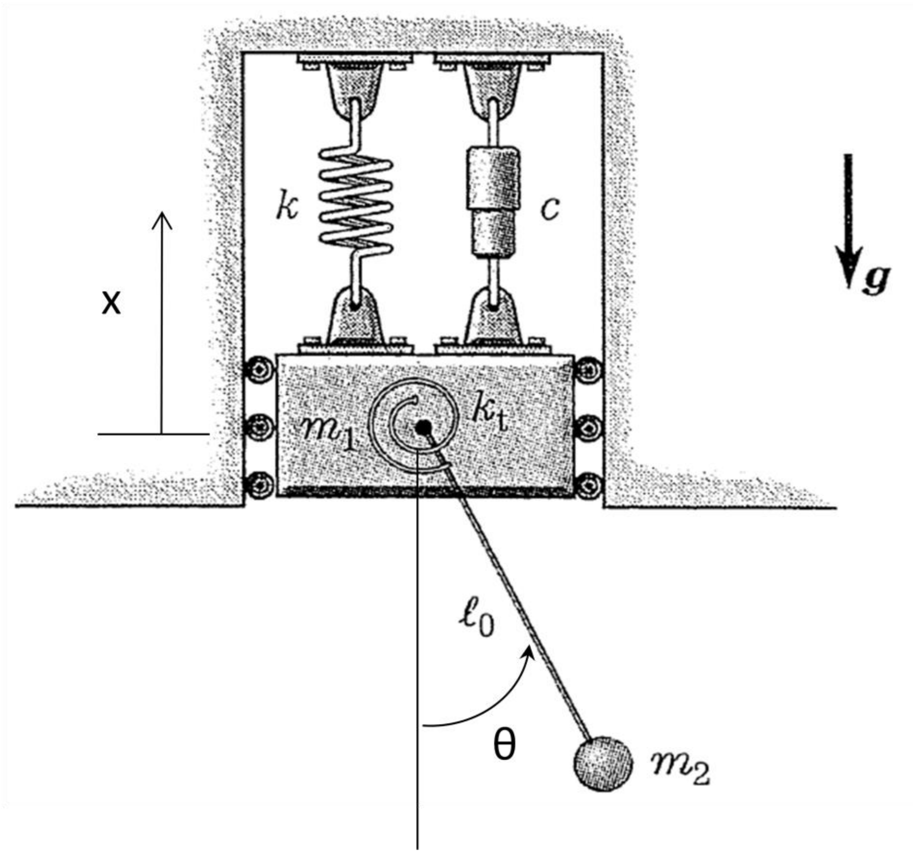
Substituting (3) *into* (1) *and solving for* $\ddot{x}$ *give*

$$\ddot{x} = -\frac{2kx}{m} - g\theta \qquad (4)$$

Second, we can choose the following state variables : $\{x, \dot{x}, \theta, \dot{\theta}\}$.
Expressing (3) *and* (4) *in state – space format,*

$$\begin{Bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-2k}{m} & 0 & g & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2k}{ml} & 0 & \frac{-2g}{l} & 0 \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{Bmatrix}$$

Problem 3 (30 points) : A mass m_1 is constrained to move in a vertical channel and is restrained by a spring of spring constant k and a dashpot of dashpot constant c. A simple pendulum of length ℓ_o and mass m_2 is suspended from m_1 as sketched below. A torsional spring of spring constant k_t restrains the rotation of the pendulum and is undeformed when the pendulum hangs vertically downward. Note that gravity acts. Use the Langrange's method to derive the equation(s) of motion for the system.



$Two\ independent\ coordinates: q_1 = x, q_2 = \theta \quad (2\ pts)$

$Q_1 = 0 \quad and \quad Q_2 = 0 \ \ (2\ pts)$

$R_x = \frac{1}{2} c\dot{x}^2 \quad and \ \ R_\theta = 0 \ \ (4\ pts)$

$U = \frac{1}{2} kx^2 + m_1 gx + \frac{1}{2} k_t \theta^2 + m_2 g l_0 (1 - \cos\theta) \ \ (8\ pts)$

$T = T_{cart} + T_{pendulum}$

$T_{cart} = \frac{1}{2} m_1 \dot{x}^2 \ \ (2\ pts)$

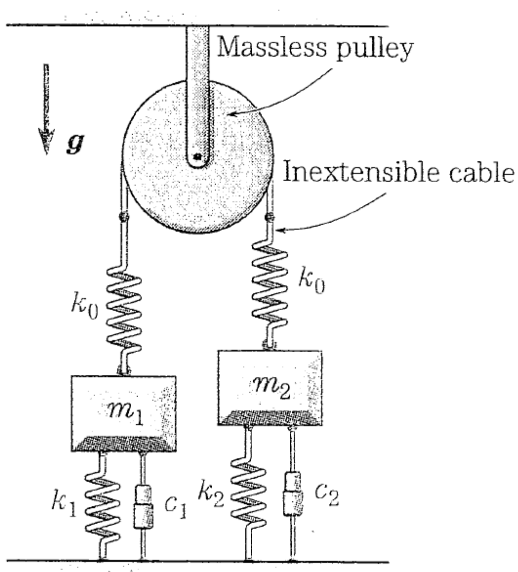
$T_{pendulum} = \frac{1}{2} m_2 \left((\dot{x} + l_0 \dot{\theta} \sin\theta)^2 + (l_0 \dot{\theta} \cos\theta)^2 \right) = \frac{1}{2} m_2 \left(\dot{x}^2 + (l_0 \dot{\theta})^2 + 2\dot{x} l_0 \dot{\theta} \sin\theta \right) \ \ (8\ pts)$

Substitute into Lagrange's Equations :

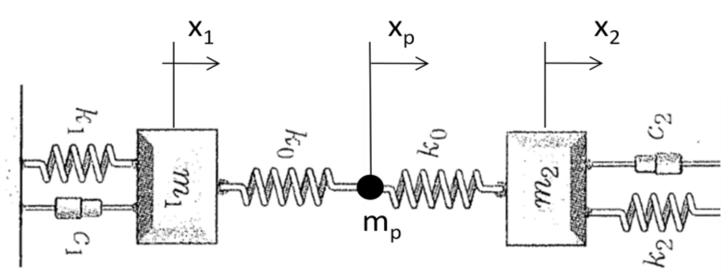
$(\mathrm{m}_1 + m_2)\ddot{x} + c\dot{x} + kx + m_1 g + m_2 l(\ddot{\theta} \sin\theta + \dot{\theta}^2 \cos\theta) = 0 \ \ (2\ pts)$

$m_2 l_0 \ddot{\theta} + k_t \theta + m_2 g l_0 \sin\theta + m_2 \ddot{x} l_0 \sin\theta = 0 \ \ (2\ pts)$

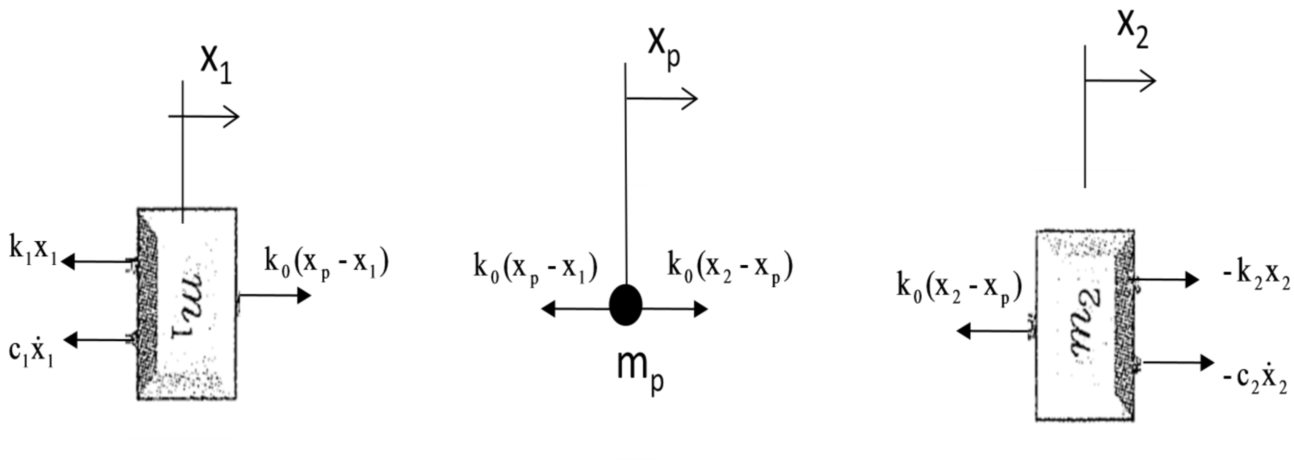
Problem 4 (30 points): Two masses m_1 and m_2 are connected by a cable that runs over a massless pulley without slip, as sketched below. The cable is extensible and thus has been modeled as a section of inextensible cable combined with a spring of spring constant k_0 connected in series at each of its ends. Each of the masses is connected to ground by a spring of spring constant k_1 or k_2 and a dashpot of dashpot constant c_1 or c_2 as shown. Note that gravity acts. Use the Newtonian method to derive the equation(s) of motion for the system.



A possible definition of the coordinate system is shown below, with x_p defined as the displacement of the pulley, and m_p ,the mass of the pulley, equals to zero.



Free body diagram for each mass: (12 pts)



Assuming that gravitational force is much less than inertial, spring, and damping forces, and using Newton's Second Law gives:

$For\ \ m_1$

$m_1\ddot{x}_1 = -k_1x_1 - c_1\dot{x}_1 + k_0(x_p - x_1)$

$\text{or}\quad m_1\ddot{x}_1 + k_1x_1 + c_1\dot{x}_1 - k_0(x_p - x_1) = 0$ (1) (4pts)

$For\ \ m_2$

$m_2\ddot{x}_2 = -k_2x_2 - c_2\dot{x}_2 - k_0(x_2 - x_p)$

$or\ \ m_2\ddot{x}_2 + k_2x_2 + c_2\dot{x}_2 + k_0(x_2 - x_p) = 0$ (2) (4pts)

$For\ \ m_p$

$m_p\ddot{x}_p = 0 = -k_0(x_p - x_1) + k_0(x_2 - x_p)$

$or\ x_p = (x_2 - x_1)/2$ (3) (4pts)

$Substitute\ \ (3)\ \text{int}\,o\ \ (1)\ \ and\ \ (2)$

$m_1\ddot{x}_1 + c_1\dot{x}_1 + (k_1 - \frac{3}{2}k_0)x_1 + \frac{1}{2}k_0x_2 = 0$ (3pts)

$m_2\ddot{x}_2 + c_2\dot{x}_2 + (k_2 + \frac{1}{2}k_0)x_2 + \frac{1}{2}k_0x_1 = 0$ (3pts)