Frequency Response

Agenda

- Preliminaries
- First order systems
  - Frequency response
  - Low-pass filter
- Second order systems
  - Classical solutions
  - Frequency response
- Higher order system

Frequency response

- Steady-state behavior of systems to harmonic excitations over a range of input frequencies
- Determination of important behavioral characteristics of dynamic systems by subjecting them to harmonic inputs and observing the response
  - Experimentally,
  - Analytically, or
  - Numerically.

Preliminaries - ODE

$$\text{Ordinary, linear, constant-coefficient differential equations}$$

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots. + a_1 \frac{dx}{dt} + a_0 x = Gu(t)$$

$$\textit{Solutions can be expressed}$$

$$x(t) = x_h(t) + x_p(t)$$

$$\textit{Homogeneous equation}$$

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots. + a_1 \frac{dx}{dt} + a_0 x = 0$$

$$\textit{The solution can be written in the form}$$

$$x_h(t) = Ce^{\lambda t}$$

$$a_n C\lambda^n e^{\lambda t} + a_{n-1} C\lambda^{n-1} e^{\lambda t} + \ldots + a_1 C\lambda e^{\lambda t} + a_0 Ce^{\lambda t} = 0$$

$$a_n \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0$$

$$\textit{This expression is called the characteristic equation}\ (CE)$$

Preliminaries - Laplace transform

$$\left[a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0\right] X(s) = GU(s)$$

$$TF(s) = \frac{X(s)}{U(s)} = \frac{G}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

$$\text{If derivatives are present in input,}$$

$$L(f(t)) = G(\beta_k s^k + \beta_{k-1} s^{k-1} + \ldots + \beta_1 s + \beta_0) U(s)$$

$$\frac{X(s)}{U(s)} = \frac{G(\beta_k s^k + \beta_{k-1} s^{k-1} + \ldots + \beta_1 s + \beta_0) U(s)}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

$$\text{where } k \le n$$

First Order Systems - Classical Solution

$$First - order\ equation$$

$$\tau\dot{x} + x = Gu(t)$$

$$Homogeneous\ equation\ is$$

$$\tau\dot{x} + x = 0$$

$$solution$$

$$x_h(t) = Ae^{\lambda t}$$

$$Substitution\ into\ CE,$$

$$\lambda = -1/\tau$$

$$For\ a\ harmonic\ excitation,$$

$$Gu(t) = Gu_0 \cos\omega t$$

$$The\ classical\ solution:$$

$$x(t) = \underbrace{Ae^{\lambda t}}_{\substack{homogeneous \\ (transient)}} + \underbrace{B\cos(\omega t + \phi)}_{\substack{particular \\ (steady\ state)}}$$

$$where$$

$$B = \frac{Gu_0}{\sqrt{1 + \omega^2\tau^2}}$$

$$\phi = \tan^{-1}(-\omega\tau)$$

$$and\ A\ is\ found\ from\ initial\ conditions$$

Frequency Response

The amplitude and the phase of the steady-state response

Let $\tau = RC$ *and* $Gu_0 = E$,

$$B = \frac{E}{\sqrt{1+\omega^2\tau^2}}$$

and

$$\phi = \tan^{-1}(-\omega\tau)$$

Normalized output response amplitude

$$\frac{\dfrac{E}{\sqrt{1+\omega^2\tau^2}}}{E} = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

Represent response amplitude on base – 10 logarithm,

$$dB = 20\log_{10}(B/Gu_0)$$

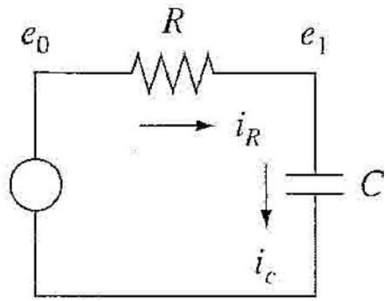
Frequency Response

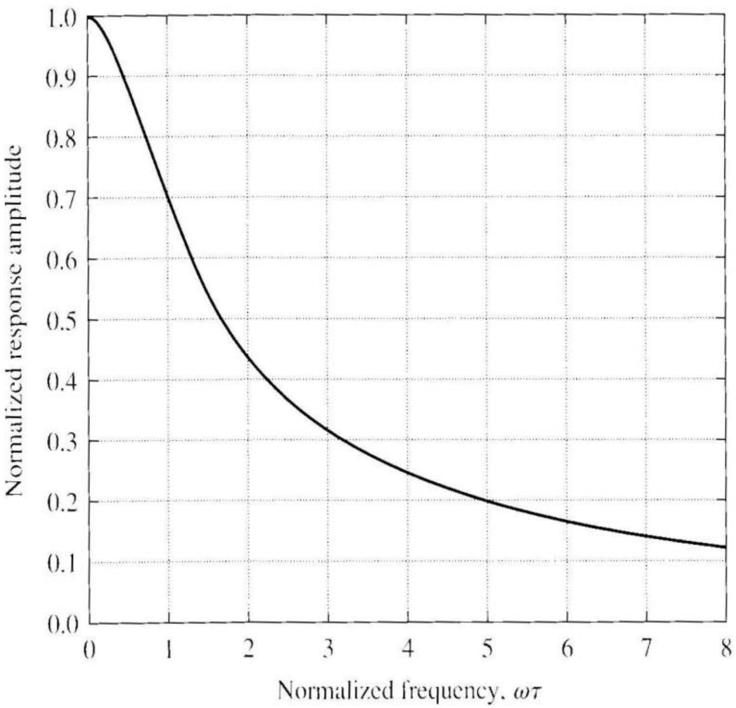
$$RC\dot{e}_1 + e_1 = e_0 = E\cos\omega t$$

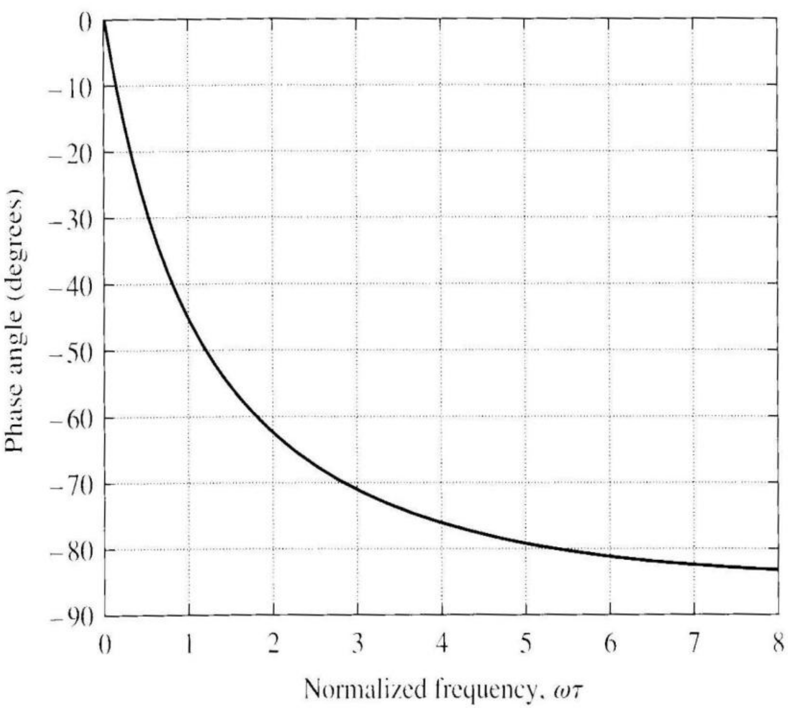
$$Let\ \tau = RC,$$

$$\tau s E_1(s) + E_1(s) = EU(s)$$

$$\frac{E_1(s)}{U(s)} = \frac{E}{s\tau + 1}$$







Frequency Response

$corner\ \ frequency = \omega\tau = 1$

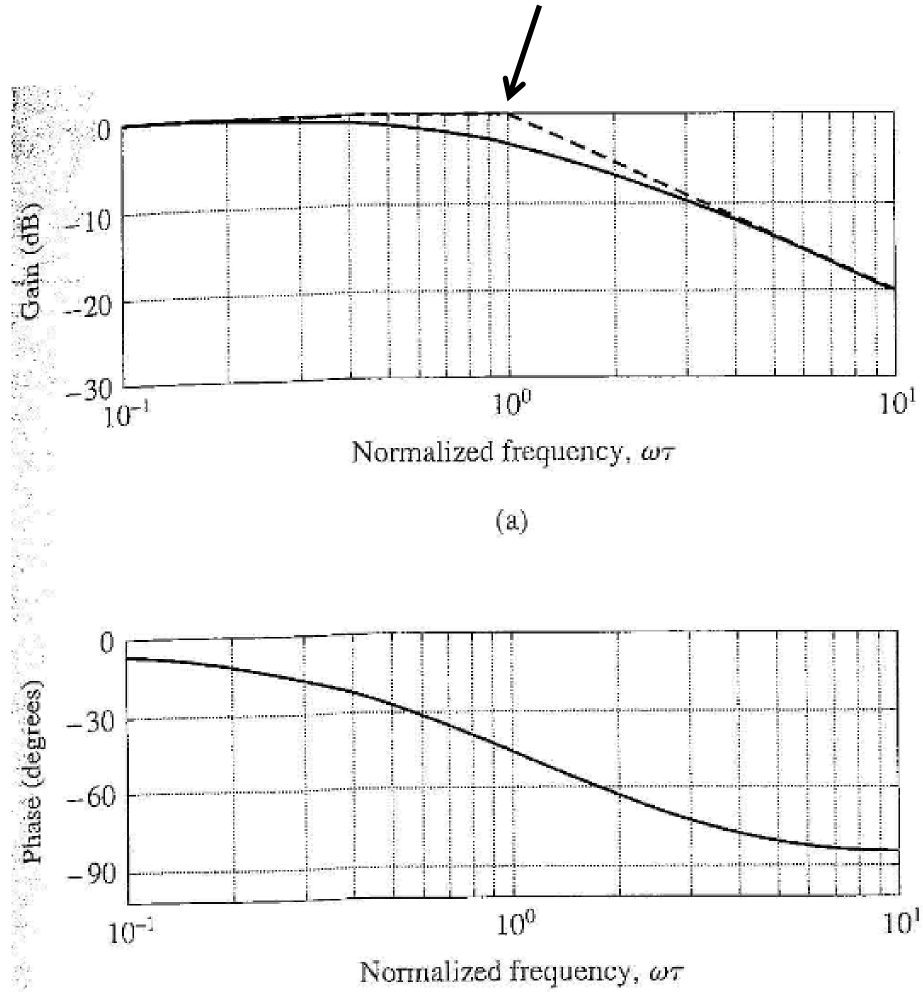
The resulting TF

$$\overline{TF} = \frac{E_1(j\omega)}{U(j\omega)} = \frac{E}{j\omega\tau + 1}$$

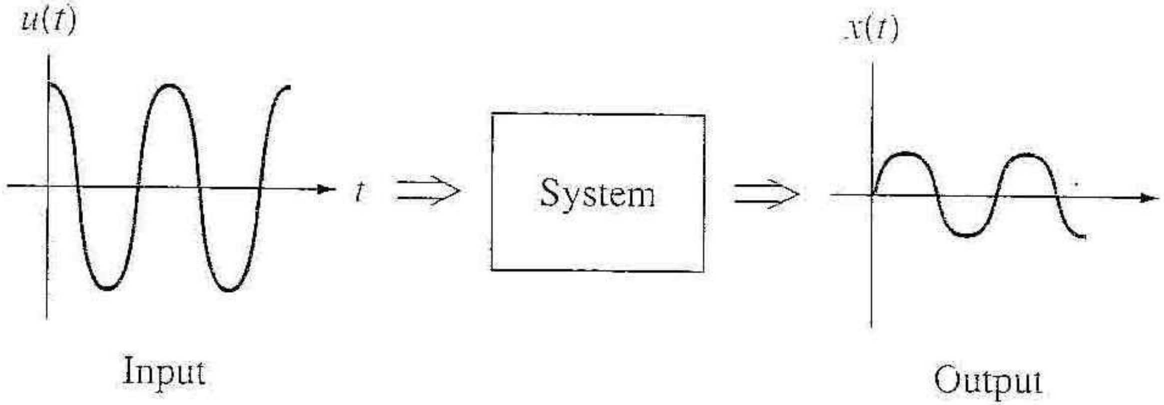
$$TF = MAG(\overline{TF}) = \frac{MAG(E)}{MAG(j\omega\tau + 1)}$$

$$TF = \frac{E}{\sqrt{1+\omega^2\tau^2}}$$

$$\phi = \tan^{-1}(-\omega\tau)$$

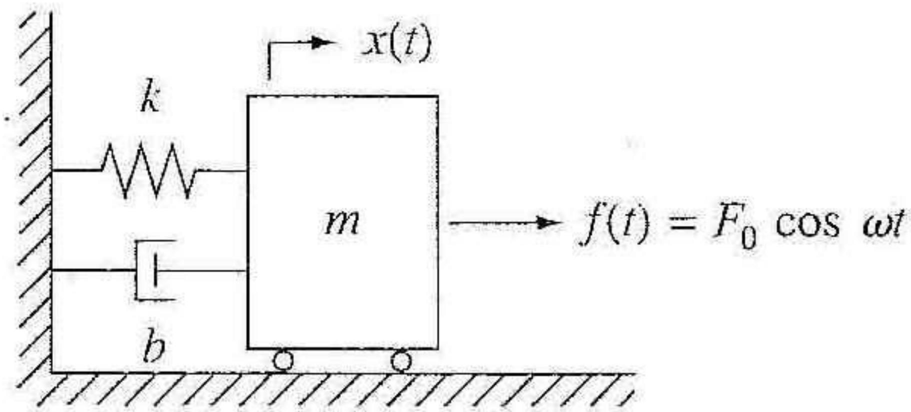


Low-Pass Filter



Typical system response

Second-order Systems



Spring-mass-damper system

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega t$$

$$\frac{m}{k}\ddot{x} + \frac{b}{k}\dot{x} + x = \frac{F_0}{k}\cos \omega t$$

$$\frac{1}{\omega_n^2}\ddot{x} + \frac{2\zeta}{\omega_n}\dot{x} + x = x_0 \cos \omega t$$

where $x_0 = F_0 / k$

The undamped natural frequency is

$$\omega_n = \sqrt{k / m}$$

damping ratio

$$\zeta = \frac{b}{b_c}$$

Second-Order Systems

Characteristic equation

$$\frac{1}{\omega_n^2}\lambda^2 + \frac{2\zeta}{\omega_n}\lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{\omega_n^2}{2}\left(-\frac{2\zeta}{\omega_n} \pm \sqrt{\frac{4\zeta^2}{\omega_n^2} - \frac{4}{\omega_n^2}}\right)$$

$$\lambda_{1,2} = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1})$$

Second-Order Systems

$$If\ \zeta\ is\ less\ than\ 1$$

$$\lambda_{1,2} = \omega_n(-\zeta \pm j\sqrt{1-\zeta^2})$$

$$x(t) = \frac{x_0 \cos(\omega t + \phi)}{\sqrt{(1-\overline{\omega}^2)^2 + (2\zeta\overline{\omega})^2}} = A\cos(\omega t + \phi)$$

$$\overline{\omega} = \omega / \omega_n$$

$$A = \frac{x_0}{\sqrt{(1-\overline{\omega}^2)^2 + (2\zeta\overline{\omega})^2}}$$

$$\phi = \tan^{-1}\left[-2\zeta\overline{\omega} /(1-\overline{\omega}^2)\right]$$

Second-Order System

- **The transfer function of a 2nd-order system:**

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- **The frequency response of this system can be modeled as:**

$$\left|H(j\omega)\right|_{dB} = -20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}$$

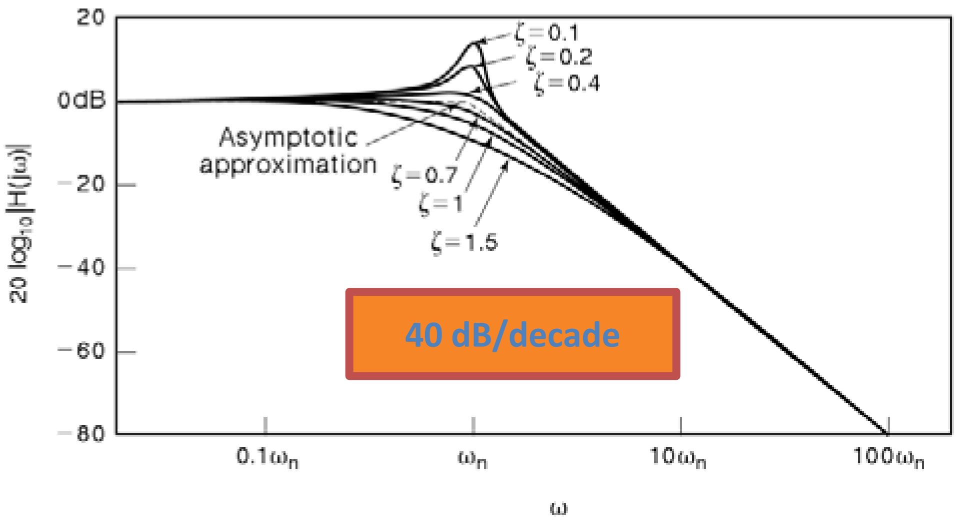
- **When** $\omega >> \omega_n$ **:**

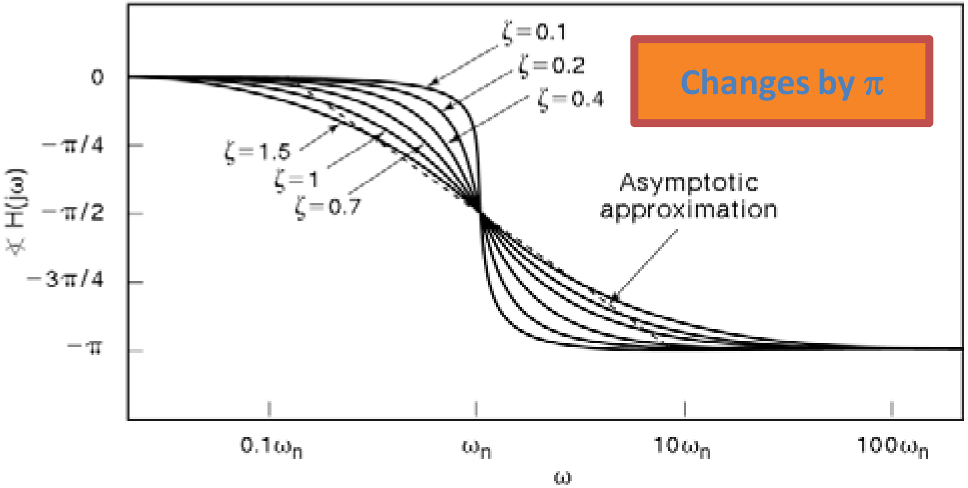
$$\left|H(j\omega)\right|_{dB} \approx -20\log\sqrt{\left(\frac{\omega^2}{\omega_n^2}\right)^2}$$

$$\approx -40\log\left(\frac{\omega}{\omega_n}\right)$$

$$\angle H(j\omega) \approx -\tan^{-1}\left(\frac{2\zeta\omega_n}{\omega}\right)$$

$$\approx 180°$$





Poles and Zeroes

- **Transfer function:**

$$H(s)=\frac{1000(s+2)}{(s+10)(s+50)} \qquad H(j\omega)=\frac{1000(j\omega+2)}{(j\omega+10)(j\omega+50)}=\frac{4\left(j\left(\frac{\omega}{2}\right)+1\right)}{\left(j\left(\frac{\omega}{10}\right)+10\right)\left(j\left(\frac{\omega}{50}\right)+1\right)}$$

- **The critical frequencies are** ω = 2 **(zero)**, 10 **(pole)**, and 50 **(pole).**

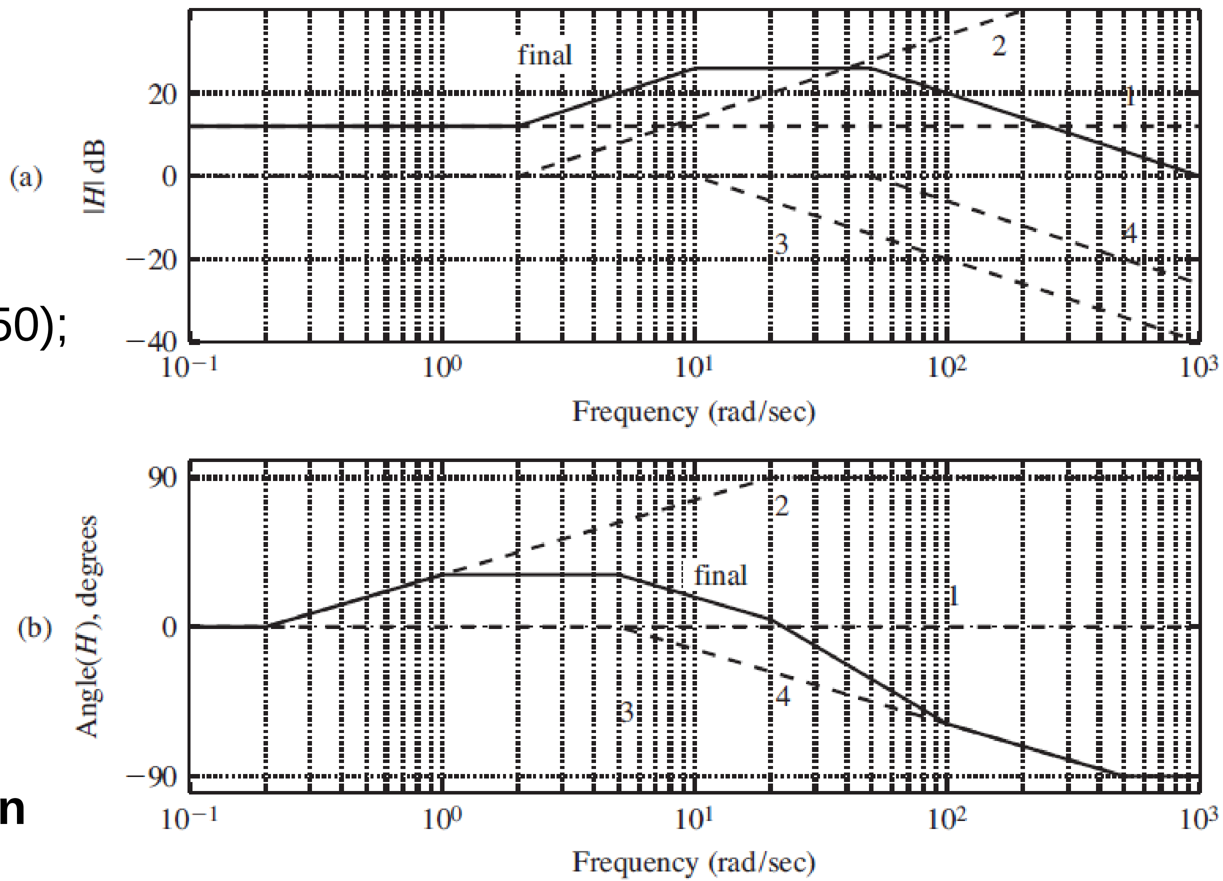
- **MATLAB (exact resp.):**

```
w = logspace(-1,3,300);
s = j*w;
H = 1000*(s+2)./(s+10)./(s+50);
magdB = 20*log10(abs(H));
phase = angle(H)*180/pi;
```

- **MATLAB (Bode):**

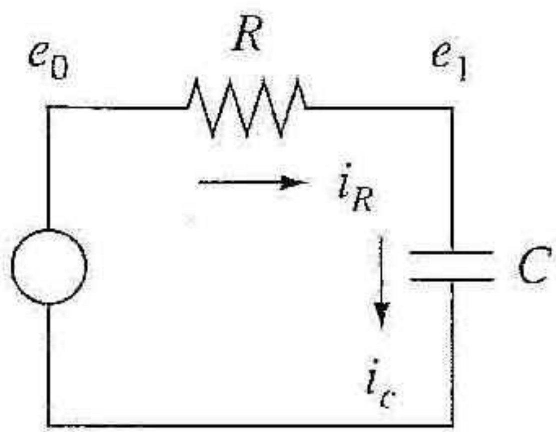
```
num = [1000 2000];
den = conv([1 1o], [1 50]);
bode(num, den);
```

- **Bode plots are useful as an analytic tool.**



Examples 1

Find the time constant for the following RC circuit



that has parameters R=8200 Ω and C=2.2μf. Sketch the amplitude frequency response and determine the corner frequency. Approximately how long does it take the transient part of the solution to this system to die out?

Solution

Given parameters: $R = 8200\,\Omega$ $C = 2.2\mu\,\text{f}$

Find the time constant, corner frequency, and transient time.

The time constant is given by Eq. (8.23) $\tau = RC = 8200$ ohm ($2.2\text{x } 10^{-6}$sec/ohm) = 0.018 sec

From Figures 8.2 & 8.3, the corner frequency is $\omega_c \tau = 1.0$

Thus the corner frequency is

$$\omega_c = \frac{1}{\tau} = \frac{1}{(8200\text{ ohm})\,(2.2 \times 10^{-6}\text{ sec / ohm})} = 55.4\text{ rad / sec}$$

All transients die out in about 4 time constants $= 4 \times \dfrac{1}{55.4\text{ rad / sec}} = 0.07\text{sec}$

Example 2

A harmonic signal of amplitude 1 and frequency 70 Hz is the input to a linear first-order system whose time constant is 0.5 second. What is the amplitude of the output? What is the phase of the output with respect to the input? Does the output lag behind the input or lead it?

Solution

Given parameters: $\tau = 0.5\,\text{sec}$ $\omega = 70\text{Hz} = (70)(2\pi)\text{rad / sec}$

Find the response amplitude and phase.

Governing equation $\tau\dot{x} + x = Gu(t) = Gu_0 \cos\omega t$

Response amplitude, Eq. (8.21), and phase, Eq. (8.22)

$$B = \frac{Gu_0}{\sqrt{1+\omega^2\tau^2}} \qquad \phi = \tan^{-1}(-\omega t)$$

$$\omega\tau = (70)(2\pi)\text{rad / sec}(0.5\,\text{sec}) = 70\pi\ \text{rad} = 219.9\,\text{rad}$$

Response amplitude: $B = \dfrac{Gu_0}{\sqrt{1+\omega^2\tau^2}} = \dfrac{1}{\sqrt{1+(219.9)^2}} = 4.55\times 10^{-3}\,\text{units}$

Phase: $\phi = \tan^{-1}(-\omega t) = \tan^{-1}(-219.9) = -89.7\,\text{deg}$

Pair-Share: Example 3

Answer the same questions as in the previous problem, except that now the system is a second-order linear system with a natural frequency of 21 rad/sec and a damping ratio of 0.25

Solution

Given parameters:
Excitation frequency: $\omega = (70)(2\pi)$ rad / sec ; Natural frequency $\omega_n = 21$ rad / sec
Damping ratio $\varsigma = 0.25$

Find the response amplitude and phase.

Eq. (8.58) Response amplitude $A = \dfrac{x_0}{\sqrt{(1-\overline{\omega}^2)^2 + (2\zeta\overline{\omega})^2}}$

Eq. (8.59) Phase angle $\phi = \tan^{-1}\dfrac{-2\zeta\overline{\omega}}{1-\overline{\omega}^2}$

Frequency ratio $\overline{\omega} = \dfrac{\omega}{\omega_n}$ $\qquad \varpi = \dfrac{2\pi(70)\,\text{rad / sec}}{21\,\text{rad / sec}} = 20.9$

Substitution gives:

Response: $A = \dfrac{1}{\sqrt{(1-(20.9)^2)^2 + (2(0.25)20.9)^2}} = \dfrac{1}{435.9} = 2.29 \times 10^{-3}$ units

Phase: $\phi = \tan^{-1}\left(\dfrac{-2(0.25)(20.9)}{1-(20.9)^2}\right) = \tan^{-1}(-0.02398) = -1.37\,\text{deg}$

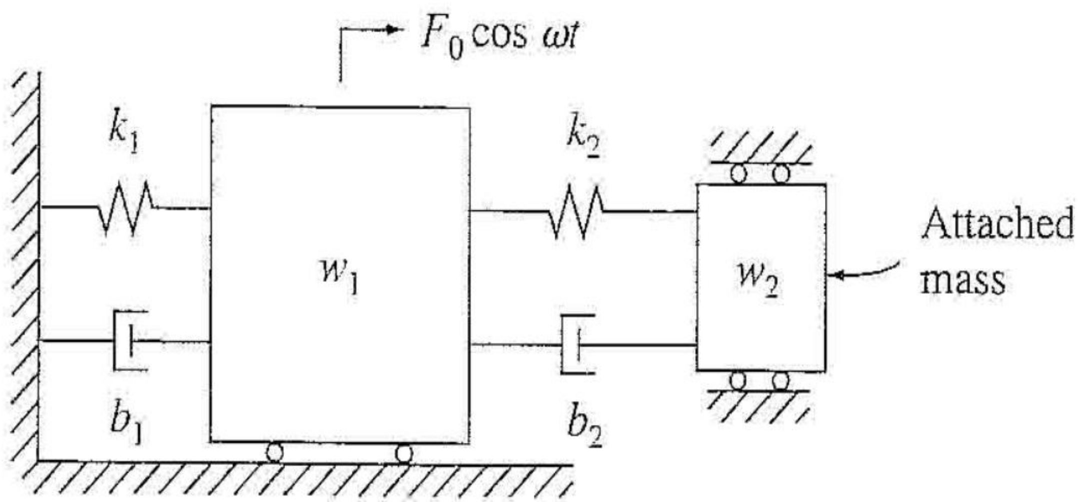
Pair-Share - Example 4

It is proposed to attach an additional spring-mass-damper system to a primary spring-mass-damper system as shown in figure below. Find the steady-state amplitude of the displacement response of the primary mass, and plot it with respect to the input frequency for the cases with an without the attached system. Such attached systems can be used to absorb unwanted vibrations. Comment on the effectiveness of reducing vibration for this system. The parameters of the problem are:

$w_1 = 200 lbf$ $\quad$ $w_2 = 50 lbf$

$k_1 = 18{,}500\, lbf\ /in$ $\quad$ $k_2 = 4600\, lbf\ /in$

$b_1 = 20\, lbf\ \sec/ in$ $\quad$ $b_2 = 5\, lbf\ \sec/ in$



Solution

Given the system shown in the figure with the parameters:

$w_1 = 200$ lbf	$k_1 = 18{,}500$ lbf/in	$b_1 = 20$ lbf sec/in
$w_2 = 50$ lbf	$k_2 = 4600$ lbf/in	$b_2 = 5$ lbf sec/in

$m_1 = 200$ lbf/386 in/sec^2 $\quad$ $m_2 = 50$ lbf/386 in/sec^2

Find the frequency response plot of x_1 with & without m_2. The equations of motion are:

$$m_1 \ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2) - b_1 \dot{x}_1 - b_2(\dot{x}_1 - \dot{x}_2) + F_0 \cos\omega t$$

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1) - b_2(\dot{x}_2 - \dot{x}_1)$$

Converting to state-space form

$$z_1 = \dot{x}_1 \ , \dot{z}_1 = \ddot{x}_1 \ , z_2 = x_1 \ , z_3 = \dot{x}_2 \ , z_4 = x_2$$

$$\dot{z}_1 = \frac{1}{m_1}[-k_1 z_2 - k_2(z_2 - z_4) - b_1 z_1 - b_2(z_1 - z_3) + F_0 \cos\omega t]$$

$$\dot{z}_2 = z_1$$

$$\dot{z}_3 = \frac{1}{m_2}[-k_2(z_4 - z_2) - b_2(z_3 - z_1)]$$

Solution (cont.)

$\dot{z}_4 = z_3$

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{Bmatrix} = \begin{bmatrix} \dfrac{-(b_1+b_2)}{m_1} & \dfrac{-(k_1+k_2)}{m_1} & \dfrac{b_2}{m_1} & \dfrac{k_2}{m_1} \\ 1 & 0 & 0 & 0 \\ \dfrac{b_2}{m_2} & \dfrac{k_2}{m_2} & -\dfrac{b_2}{m_2} & -\dfrac{k_2}{m_2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} + \begin{Bmatrix} F_0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \cos\omega t$$

Or

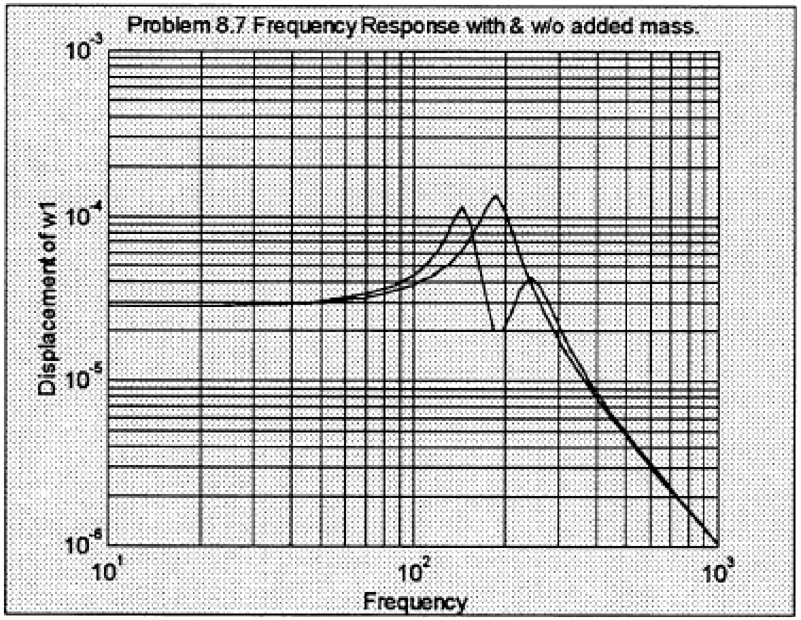
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

Without the added mass , use

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = \begin{bmatrix} -\dfrac{b_1}{m_1} & -\dfrac{k_1}{m_1} \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} \cos\omega t$$

Solution (cont.)

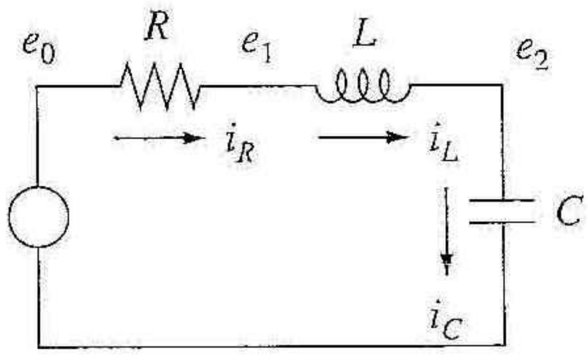
Using the MATLAB function `bode(a,b,c,d)` for the two systems described above:



The added mass shifts the peak response point down in frequency, adds an additional peak response point to the system at a higher frequency, and dramatically reduces the response at the natural frequency of the original system (without the added mass). This comparison shows how addition of a secondary mass can be used to eliminate a troublesome resonance condition. This approach is effective when the excitation frequency is constant or nearly so.

Example 5

Determine the transfer function relating the output voltage e_2 to the input voltage e_0 for the following RLC circuit:



Develop the corresponding Bode plot using data from Example 4.3. What is the natural frequency of the system? Redesign the circuit so as to increase its natural frequency by 30 percent, but keep the damping ratio at 0.707.

Solution (cont.)

Natural frequency: $\omega_n^2 = \frac{1}{LC} = 10^8$ $\omega_n = 10{,}000 = 10^4$ rad/sec

Damping ratio: $2\zeta\omega_n = \frac{RC}{LC}$ $\zeta = \frac{1}{2}\frac{R}{L}\sqrt{\frac{LC}{1}} = \frac{1}{2}R\sqrt{\frac{C}{L}} = 0.707$

New design with increased frequency:

$$\omega_{1n} = 1.3 \times 10{,}000 \text{ rad / sec} = 13{,}000 \text{ rad / sec} = \sqrt{\frac{1}{L_1 C_1}}$$

Thus $\frac{1}{L_1 C_1} = 1.69 \times 10^8 (1/\sec^2)$

If we keep C unchanged:

$$L_1 = \frac{1}{(1.69 \times 10^8 / \sec^2)(C_1 \sec / \text{ohm})} = \frac{1}{(1.69 \times 10^8 / \sec^2)(10 \times 10^{-6} \sec / \text{ohm})}$$

$$= \frac{1}{(16.9 \times 10^2)} \text{ ohm sec} = 0.00059 \text{ ohm sec}$$

Which gives: $L_1 = 0.59$ mh ; use 0.6 mh

Solution

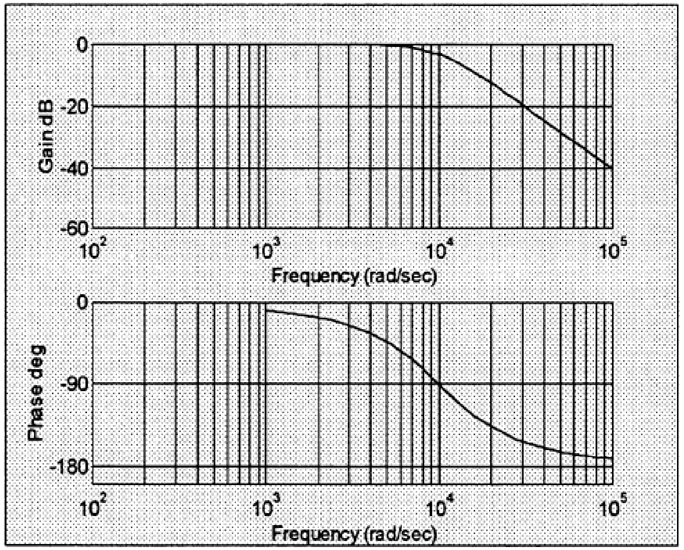
Given parameters: $L = 1$ mh, $C = 10$ μf, $R = 14.14\ \Omega$

From Eq. (4.110) $$\left[LCD^2 + RCD + 1\right]e_2 = e_0$$

$$LC = (10^{-3}\text{ ohm sec})(10^{-5}\text{ sec / ohm}) = 10^{-8}\text{ sec}^2$$

$$RC = (14.14\text{ ohm})(10^{-5}\text{ sec / ohm}) = 1.414 \times 10^{-4}\text{ sec}$$

The transfer function is: $$\frac{E_2}{E_0} = \frac{1}{10^{-8}s^2 + 1.414x10^{-4}s + 1}$$ With Bode plot as shown below



Solution (cont.)

Then $$\frac{1}{2}R_1\sqrt{\frac{C_1}{L_1}} = 0.707 \quad ; \quad R_1 = 4(0.707)\sqrt{\frac{L_1}{C_1}}$$

Or $$R_1 = 2(0.707)\sqrt{\frac{0.6\times10^{-3}\,\text{ohm sec}}{10\times10^{-6}\,\text{sec / ohm}}} = (1.414)\sqrt{0.06\times10^{3}\,\text{ohm}^2} = 10.95\,\Omega$$

The final design, of course, will normally need to incorporate commercially available components.

Example 6

Find the roots of the characteristic equation of the following systems. What are the time constants and/or natural frequencies? What are the damping ratios? Comment on the stability of each. Solve by and check by digital computation.

$$a.\ 2\ddot{x}+8\dot{x}+32x=15u(t)$$

$$b.\ \ddot{x}+20\dot{x}+25=134u(t)$$

$$c.\ 7\dddot{z}+5\ddot{z}+2\dot{z}+3z=2.5f(t) \quad <= Pair-Share$$

$$d.\ \dddot{z}+2\ddot{z}+4\dot{z}+5z=4f(t) \quad <= Pair-Share$$

Solution

$a.\ 2\ddot{x}+8\dot{x}+32x=15u(t)$

Let $x_h = Ce^{\lambda t}$ Substituting gives: $(2\lambda^2+8\lambda+32)Ce^{\lambda t}=0$

From which: $\lambda_{1,2}=\frac{1}{4}\left(-8\pm\sqrt{8^2-4(2)(32)}\right)=-2\pm j\frac{1}{4}\sqrt{192}=a+jb$

Using Eqs. (8.94) (8.95): Natural frequency: $\omega_n=\sqrt{a^2+b^2}=4$ rad / sec ;

Damping ratio $\zeta=\frac{-a}{\omega_n}=0.5$

$b.\ \ddot{x}+20\dot{x}+25=134u(t)$

$\lambda_1=-1.34$ $\lambda_2=-18.6$ $\tau_1=0.746$ sec $\tau_2=0.0538$ sec

Solution (cont.)

$$c.\ 7\dddot{z} + 5\ddot{z} + 2\dot{z} + 3z = 2.5 f(t)$$

$\lambda_{1,2} = +0.1 \pm j0.677$ $\quad \zeta = -0.146 \quad \omega_n = 0.685$

$\lambda_3 = -0.9144$ $\quad \tau_3 = 1.094$ sec

This system is unstable.

$$d.\ \dddot{z} + 2\ddot{z} + 4\dot{z} + 5z = 4 f(t)$$

$\lambda_{1,2} = -0.237 \pm j1.795$ $\quad \zeta = 0.131 \quad \omega_n = 1.81$ rad / sec

$\lambda_3 = -1.526$ $\quad \tau_3 = 0.655$ sec

Example 7 Problem 8.12 a,c,e,g

Find the transfer function for each of the systems below, relating the output x or z to the input u or f.

$$a.\ 2\ddot{x} + 8\dot{x} + 32x = 15u(t)$$

$$b.\ \ddot{x} + 20\dot{x} + 25 = 134u(t)$$

$$c.\ 100\ddot{x} + 400\dot{x} + 1 = u(t) \qquad <= Pair - Share$$

$$d.\ 5\ddot{z} + 2\dot{z} + 4z - 3\dot{v} - 4v = 8f(t) \qquad <= Pair - Share$$

$$\ddot{v} + 3\dot{v} + 12v - 3z = 0$$

Solution

Take Laplace transform of each equation and solve for the response variable divided by the input variable. For d, first put into classical form.

a. $2\ddot{x}+8\dot{x}+32x=15u(t)$

$$\frac{15}{2s^2+8s+32}$$

b. $\ddot{x}+20\dot{x}+25=134u(t)$

$$\frac{134}{s^2+20s+25}$$

c. $100\ddot{x}+400\dot{x}+1=u(t)$

$$\frac{1}{100s^2+400s+1}$$

d. $5\ddot{z}+2\dot{z}+4z-3\dot{v}-4v=8f(t)$

$\ddot{v}+3\dot{v}+12v-3z=0$

$$\frac{8s^2+24s+96}{5s^4+17s^3+70s^2+27s+36}$$

Example 8

Find the set of state-space equations for each of the below systems:

$$a.\ 2\ddot{x}+8\dot{x}+32x=15u(t)$$

$$b.\ \ddot{x}+20\dot{x}+25=134u(t)$$

$$c.\ 100\ddot{x}+400\dot{x}+1=u(t)$$

$$d.\ 5\ddot{z}+2\dot{z}+4z-3\dot{v}-4v=8f(t)$$

$$\ddot{v}+3\dot{v}+12v-3z=0$$

Solution

For a linear, constant coefficient, second order equation we have:

$$a\ddot{x} + b\dot{x} + cx = Gu(t)$$

let $x_1 = \dot{x}$ then $\dot{x}_1 = \ddot{x}$ and $x_2 = x$ $\dot{x}_2 = \dot{x}$

Substitution gives:

$$a\,\dot{x}_1 + b\,x_1 + cx_2 = Gu(t)$$

Or

$$\dot{x}_1 = \frac{[Gu(t) - b\,x_1 - cx_2]}{a}$$

$$\dot{x}_2 = x_1$$

Solution (cont.)

Using this procedure

a. $a = 2, \quad b = 8, \quad c = 32, \quad G = 15$

$$\dot{x}_1 = \frac{[15u(t) - 8x_1 - 32x_2]}{2}$$

$$\dot{x}_1 = [7.5u(t) - 4x_1 - 16x_2]$$

$$\dot{x}_2 = x_1$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -4 & -16 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 7.5 \\ 0 \end{pmatrix} u(t)$$

Alternatively use the MATLAB procedure `tf2ss` to convert from transfer function form to the following state-space form.

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

Using `tf2ss` for this example gives:

$$\mathbf{A} = \begin{bmatrix} -4 & -16 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{C} = [0 \quad 7.5] \qquad \mathbf{D} = 0$$

Solution (Cont)

The remaining problems are solved using the MATLAB function `tf2ss`.

$$b.\ \ddot{x} + 20\dot{x} + 25 = 134u(t)$$

$$\mathbf{A} = \begin{bmatrix} -20 & -25 \\ 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathbf{C} = [0 \quad 134] \quad \mathbf{D} = 0$$

$$c.\ 100\ddot{x} + 400\dot{x} + 1 = u(t)$$

$$\mathbf{A} = \begin{bmatrix} -4 & -0.01 \\ 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathbf{C} = [0 \quad 0.01] \quad \mathbf{D} = 0$$

Solution (Cont)

$$d.\ 5\ddot{z}+2\dot{z}+4z-3\dot{v}-4v=8f(t)$$

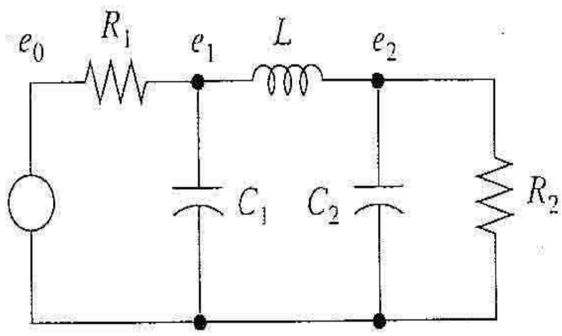
$$\ddot{v}+3\dot{v}+12v-3z=0$$

$$\mathbf{A}=\begin{bmatrix}-3.4 & -14.0 & -5.4 & -7.2\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix} \quad \mathbf{B}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \mathbf{C}=\begin{bmatrix}0 & 1.6 & 4.8 & 19.2\end{bmatrix}$$

$$\mathbf{D}=0$$

Example 9

Determine the transfer function coefficients for the circuit :



If:

$R_1 = 10\,\Omega \qquad R_2 = 100\,\Omega$

$C_1 = 1\,\mu f \qquad C_2 = 22\,\mu f$

$L = 10\,mh$

Find the eigen values, and check the stability of the system. If the system is stable, determine its Bode plot.

Solution

From Problem 4.14, the transfer function can be stated as follows.

$$\frac{e_2}{e_0} = \frac{1}{R_1C_1LC_2D^3 + \left(LC_2 + \frac{R_1}{R_2}LC_1\right)D^2 + \left(R_1C_1 + R_2C_2 + \frac{L}{R_2}\right)D + \left(1 + \frac{R_1}{R_2}\right)}$$

Or, in normalized form,

$$\frac{e_2}{e_0} = \frac{\frac{1}{(1+R_1/R_2)}}{\frac{R_1C_1LC_2}{(1+R_1/R_2)}D^3 + \frac{\left(LC_2 + \frac{R_1}{R_2}LC_1\right)}{(1+R_1/R_2)}D^2 + \frac{\left(R_1C_1 + R_2C_2 + \frac{L}{R_2}\right)}{(1+R_1/R_2)}D + 1}$$

Using the component values stated, $\frac{R_1}{R_2} = \frac{10\,\text{ohm}}{100\,\text{ohm}} = 0.1$

$$R_1\ C_1 = 10\text{ ohm} \times 1 \times 10^{-6}\ \frac{\text{sec}}{\text{ohm}} = 10 \times 10^{-6}\text{ sec}$$

Solution (cont.)

$$R_2\, C_2 \;=\; 100 \text{ ohm} \times 10 \times 10^{-6} \,\frac{\text{sec}}{\text{ohm}} \;=\; 1000 \times 10^{-6} \text{ sec}$$

$$\frac{L}{R_2} \;=\; \frac{0.005 \text{ ohm sec}}{100 \text{ ohm}} \;=\; 50 \times 10^{-6} \text{ sec}$$

$$L\, C_1 \;=\; 0.005 \text{ ohm sec} \times 1 \times 10^{-6} \,\frac{\text{sec}}{\text{ohm}} \;=\; 0.005 \times 10^{-6} \text{ sec}^2$$

$$L\, C_2 \;=\; 0.005 \text{ ohm sec} \times 10 \times 10^{-6} \,\frac{\text{sec}}{\text{ohm}} \;=\; 0.050 \times 10^{-6} \text{ sec}^2$$

Thus the transfer function becomes

$$\frac{e_2}{e_0} \;=\; \frac{0.9091}{0.45455 \times 10^{-12} D^3 + 0.0459091 \times 10^{-6} D^2 + 963.636 \times 10^{-6} D + 1}$$

Solution (cont.)

The Bode plot is