Mixed Discipline Systems

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ME584

Agenda

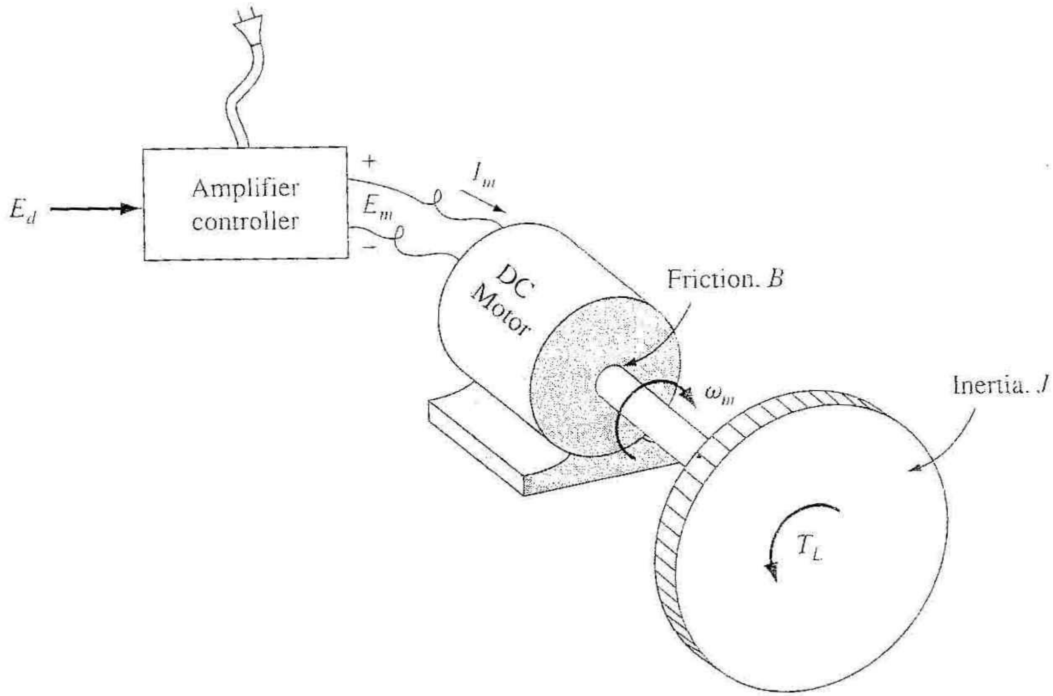
- Electro-mechanical Systems
  - DC Motor Speed Control
  - DC Motor Position Control
- Fluid-Mechanical Systems
  - Hydraulic Position Servo
  - Electro-hydraulic Position Servo
  - Pneumatic Position Servo

- Active Learning: Pair-share Exercises

Electro-mechanical Systems

DC Motor Speed Control (Open-Loop Control)

Without Feedback



Open-loop DC motor speed control

DC Motor Speed Control (Open-Loop Control)

The model for the motor is based on ideal permanent-magnet DC motor

$$T_m = K_t I_m$$

$$E_m = K_v \omega_m$$

$$K_t = K_v = K$$

The motor is driven by an electronic power source, modeled as ideal voltage source driving motor through resistance R_0 . Assuming overall amplifier gain G:

$$E_m = GE_d - R_0 I_m$$

DC Motor Speed Control (Open-Loop Control)

$Torque\ balance:$

$$T_m - J\dot{\omega}_m - B\omega_m - T_L = 0$$

$Solve\ this\ equation\ for\ speed$

$$J\dot{\omega}_m + B\omega_m = T_m - T_L = K\left[\frac{GE_d - (K\omega_m)}{R_0}\right] - T_L$$

$$\left[JD + B + \frac{K^2}{R_0}\right]\omega_m = \frac{GK}{R_0}E_d - T_L$$

$Transfer\ function\ is\ a\ first-order\ function\ of\ input\ E_d\ and\ disturbance\ T_L$

$$\omega_m = \frac{\dfrac{G/K}{(1+\dfrac{B}{K^2/R_0})}E_d - \dfrac{R_0/K^2}{(1+\dfrac{B}{K^2/R_0})}T_L}{\dfrac{\dfrac{J}{K^2/R_0}}{1+\dfrac{B}{K^2/R_0}}D + 1}$$

DC Motor Speed Control (Open-Loop Control)

$$Static\ \ gain = \left.\frac{\partial \omega_m}{\partial E_d}\right|_{D=0} = \frac{G/K}{(1+\dfrac{B}{K^2/R_0})}$$

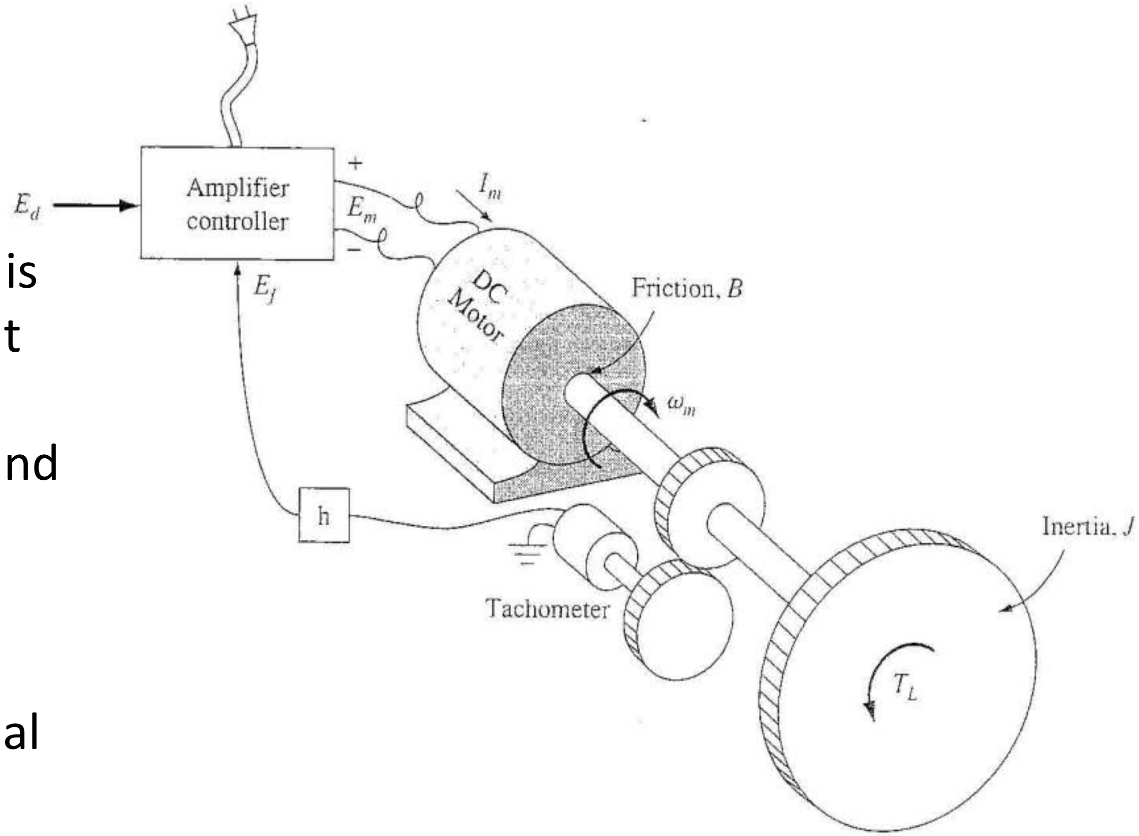
$$Disturbance\ sensitivity = \left.\frac{\partial \omega_m}{\partial T_L}\right|_{D=0} = -\frac{R_0/K^2}{(1+\dfrac{B}{K^2/R_0})}$$

$$Time\ constant\ = \tau = \frac{\dfrac{J}{K^2/R_0}}{(1+\dfrac{B}{K^2/R_0})}$$

DC Motor Speed Control (Closed-Loop Control)

With Speed Feedback

- Purpose of feedback control is to allow output to track input and to compensate for any error from command input and the actual output.
- Summing junction subtracts feedback signal from input signal to obtain an error signal which is usually amplified to drive actuator.
- Gain of tachometer is h



DC Motor Speed Control

$$E_f = h\omega_m$$

$$T_m = KI_m$$

$$E_m = K\omega_m$$

The equation for amplifier

$$E_m = Ge - R_0 I_m$$

where

$$e = E_d - E_f = error$$

$$E_d = \text{voltage representing the desired speed}$$

DC Motor Speed Control

$$Torque\ balance\ equation$$

$$T_m - J\dot{\omega}_m - B\omega_m - T_L = 0$$

$$Solving\ this\ equation\ for\ speed$$

$$J\dot{\omega}_m + B\omega_m = T_m - T_L = K\left[\frac{G(E_d - h\omega_m) - (K\omega_m)}{R_0}\right] - T_L$$

$$\left[JD + B + \frac{K^2}{R_0} + \frac{GKh}{R_0}\right]\omega_m = \frac{GK}{R_0}E_d - T_L$$

DC Motor Speed Control

Transfer function is first-order function of input and disturbance

$$\omega_m = \frac{\dfrac{1/h}{\left(1+\left(\dfrac{1+\dfrac{B}{K^2/R_0}}{\dfrac{Gh}{K}}\right)\right)}E_d - \dfrac{R_0/K^2}{\left(1+\dfrac{B}{K^2/R_0}+\dfrac{Gh}{K}\right)}T_L}{\dfrac{\dfrac{J}{K^2/R_0}}{\left(1+\dfrac{B}{K^2/R_0}+\dfrac{Gh}{K}\right)}D+1}$$

DC Motor Speed Control

$$Static\ gain = \left.\frac{\partial \omega_m}{\partial E_d}\right|_{D=0} = \frac{G/K}{\left(1 + \frac{B}{K^2/R_0} + \frac{Gh}{K}\right)}$$

$$Disturbance\ sensitivity = \left.\frac{\partial \omega_m}{\partial T_L}\right|_{D=0} = -\frac{R_0/K^2}{\left(1 + \frac{B}{K^2/R_0} + \frac{Gh}{K}\right)}$$

$$Time\ constant = \tau = \frac{J/(K^2/R_0)}{\left(1 + \frac{B}{K^2/R_0} + \frac{Gh}{K}\right)}$$

Static gain, disturbance sensitivity, time constant of closed-loop system are modified by

$$(1 + B/(K^2/R_0) + (Gh)/K)$$

A reduction in static gain, not desirable, can be compensated by increasing input. A reduction in disturbance sensitivity means motor speed will not vary as much. A reduction in time constant means system will respond faster.

DC Motor Position Control

The motor equations:

$$T_m = KI_m$$

$$E_m = K\omega_m$$

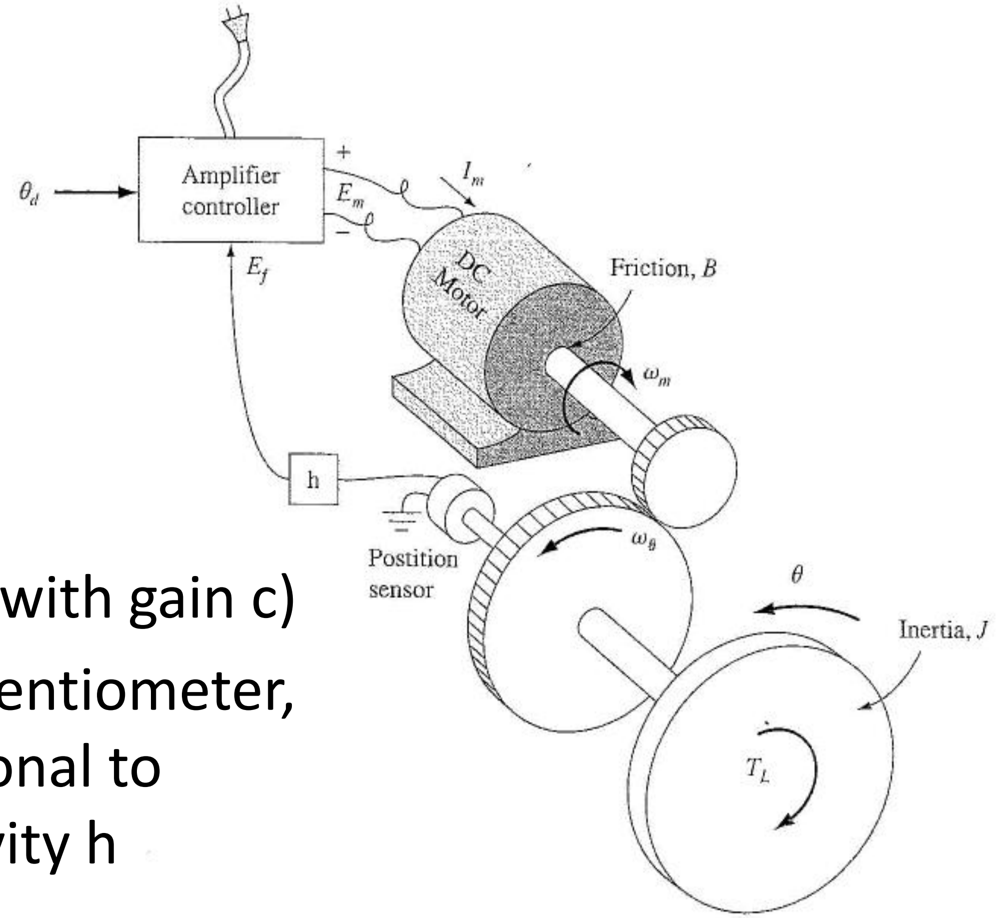
The equation for amplifier:

$$E_m = G(c\theta_d - E_f) - R_0 I_m$$

Set point/desired position is θ_d (with gain c)

Feed back position sensor is potentiometer, output voltage linearly proportional to angular position, θ, with sensitivity h

$$E_f = h\theta$$



Position control system using DC motor

DC Motor Position Control

The motor speed ω_m is reduced to an output shaft ω_θ by a gear speed ration R_s

$$\omega_\theta = R_s \omega_m$$

$$T_\theta = \frac{1}{R_s} T_m$$

$$\omega_\theta = \dot{\theta}$$

Neglecting inertia of gear train, assuming viscous damping B in the drive train, torque balance gives

$$T_\theta - J\ddot{\theta} - B\dot{\theta} - T_L = 0$$

DC Motor Position Control

Combining preceding equations:

$$J\ddot{\theta} + B\dot{\theta} = \frac{T_m}{R_s} - T_L = \frac{1}{R_s}\left\{K\left[\frac{G(c\theta_d - h\theta) - (K\dfrac{D\theta}{R_s})}{R_o}\right]\right\} - T_L$$

$$\left[JD^2 + (B + \frac{K^2}{R_0 R_s^2})D + \frac{GKh}{R_0 R_s}\right]\theta = \frac{GK_c}{R_0 R_s}\theta_d - T_L$$

Transfer function is a second − order function of input and disturbance

$$\theta = \frac{\dfrac{c}{h}\theta_d - \dfrac{1}{\left(\dfrac{GKh}{R_0 R_s}\right)}T_L}{\dfrac{J}{\left(\dfrac{GKh}{R_0 R_s}\right)}D^2 + \dfrac{(B + K^2/(R_0 R_s^2))}{\left(\dfrac{GKh}{R_0 R_s}\right)}D + 1}$$

DC Motor Position Control

$$\text{Static gain}\ \ G_s = \left.\frac{\partial \theta}{\partial \theta_d}\right|_{D=0} = \frac{c}{h}$$

Sensitivity of position to variation in torque is inverse of stiffness: effect of position feedback is to create artificial spring.

Static stiffness:

$$k_s = -\left.\frac{\partial T_L}{\partial \theta}\right|_{D=0} = \frac{GKh}{R_0 R_s}$$

Dynamic characteristics:

$$\omega_n = \sqrt{\frac{\left(\frac{GKh}{R_0 R_s}\right)}{J}},\ \ \zeta = \frac{B + K^2/(R_0 R_s^2)}{2\sqrt{J\left(\frac{GKh}{R_0 R_s}\right)}}$$

General Approach to Model Electro-Mechanical Systems

1. Apply motor and amplifier equations

   $T_m = K i_m$

   $E_m = K \omega_m$

   $E_m = Ge - R_0 i_m$

2. Write force or torque balance equation

   $F = ma$ or $T = J\alpha$

3. Combine equations to obtain transfer function, and compute static and dynamic characteristics

Example 1

A DC motor is connected to 125 volts DC and is used to spin a grindstone. The diameter of the grindstone disk is 200 mm, and the thickness is 15 mm. The density of the disk material is 3000 kg/m^3 . The free speed of the motor (with maximum voltage applied and no torque loading) is 1000 rpm. The voltage source that is used to drive the motor has an output impedance R_0 of 2 Ω (i.e.,$e_m=e_0-R_0 i_m$). Neglecting the inertia and friction of the motor itself, derive a transfer function for the dynamic response of the speed of the motor as a function of the input voltage. Calculate the time response using Laplace transform techniques for a step input of 125 volts DC with zero initial conditions.

Example 1

The torque from an ideal DC motor is $$T_m = K_m\, i_m$$

Speed of an ideal motor requires voltage. $$e_m = K_m\, \omega_m$$

The torque balance on the motor shaft includes the torque generated by the motor and the inertia of the grindstone.

$$T_m = J\, \dot{\omega}_m$$

The voltage source driving the motor has an output impedance of R_o and can be modeled as follows.

$$e_m = e_o - R_o\, i_m$$

Substituting the ideal motor equations into the above voltage equations, and using the torque equation yields the following.

$$K_m\, \omega_m = e_o - R_o \frac{T_m}{K_m} = e_o - R_o \frac{J\, \dot{\omega}_m}{K_m}$$

Solving for speed

$$\omega_m = \frac{\dfrac{e_o}{K_m}}{\dfrac{J\, R_o}{K_m^2} D + 1} = \frac{G_s\, e_o}{\tau D + 1}$$

Example 1

The static gain is $G_s = \frac{1}{K_m}$ and the time constant is $\tau = \frac{J\,R_o}{K_m^2}$.

The values for the static gain and the time constant can be calculated from the performance information given. The mass of the grindstone is

$$m = \rho V = \rho \frac{\pi d^2}{4} t = 3000 \frac{\text{kg}}{\text{m}^3} \frac{\pi 0.2^2\ \text{m}^2}{4} 0.015\,\text{m} = 1.41\,\text{kg}$$

The rotational inertia is $J = m\frac{d^2}{8} = 1.41\,\text{kg} \frac{0.2^2\ \text{m}^2}{8} = 0.00707 \frac{\text{kg m}^2}{\text{s}}$

The motor constant is $K_m = \frac{e_{\max}}{\omega_{\max}} = \frac{125\ \text{volts}}{1000\ \frac{\text{rev}}{60\ \text{s}}} = 7.5 \frac{\text{volts}\ s}{\text{rev}\ \frac{2\pi\,\text{rad}}{\text{rev}}} = 1.19 \frac{\text{joule}}{\text{amp}}$

Using the output impedance as 2 ohms, we can calculate the time constant.

Example 1

$$\tau = \frac{J\,R_o}{K_m^2} = \frac{0.00707\,\frac{\text{kg m}^2}{\text{s}}\,2\,\frac{\text{volt}}{\text{amp}}}{1.19^2\left(\frac{\text{joule}}{\text{amp}}\right)^2} = 0.00992\,\frac{\frac{\text{N m}}{\text{s}}\,\text{s}^2\,\frac{\text{volt}}{\text{amp}}\,\text{amp}^2}{\text{joule}^2}$$

$$\tau = 9.92 \text{ ms}$$

The static gain is $G_s = \frac{1}{K_m} = 8\,\frac{\text{rpm}}{\text{volt}}$

The general time response from the Laplace transform of a first-order system with a step input is given by the following.

$$\omega_m = G_s\,u_0\left[1 - e^{-\frac{t}{\tau}}\right]$$

For this problem using $u_o = 125$ volts,

$$\omega_m = 1000\text{ rpm}\left[1 - e^{-\frac{t}{\tau}}\right] \qquad \text{where} \quad \tau = 9.92\text{ ms}$$

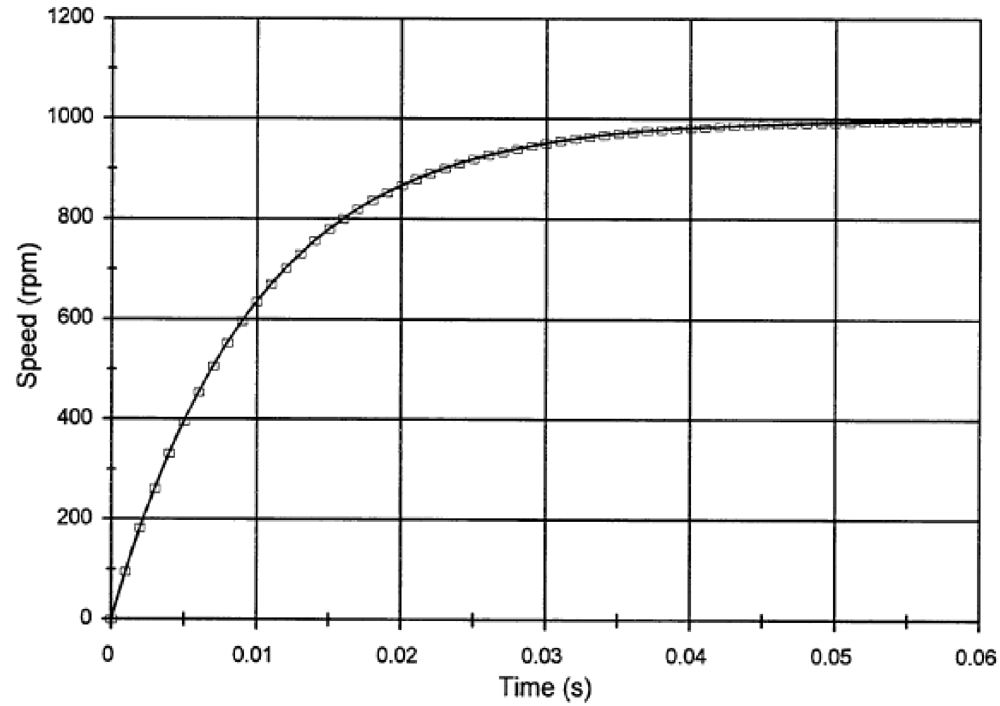
Example 2

Simulate the system with a step input of 125 V with zero initial condition

$$\omega_m = \frac{G_s e_0}{\tau D + 1}; \quad G_s = 8\frac{rpm}{V}; \tau = 9.92e-3s$$

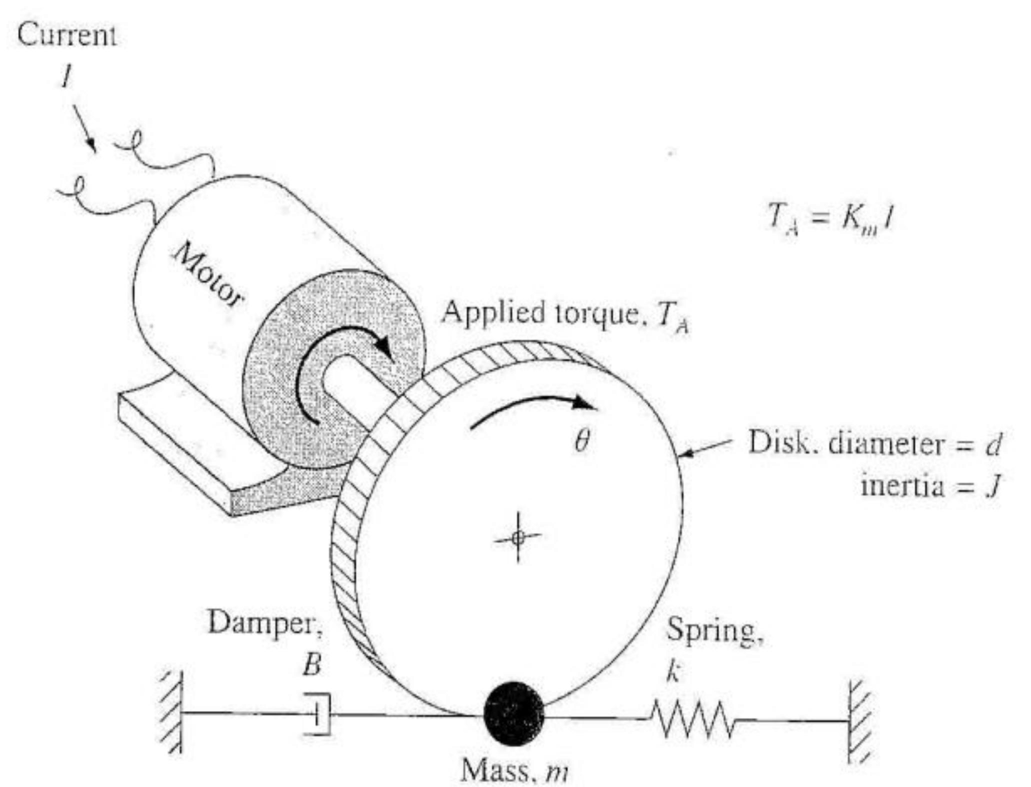
Matlab code:

```
num = [8*125];
den=[9.92e-3  1];
sys=tf(num,den);
% t = ?
t= [0:.0001:.06];
[y,t]=step(sys,t);
plot(t,y); xlabel('Time (s)');
ylabel('Speed (rpm)')
```



Pair-Share Exercise: Example 3

Shown is a DC torque motor connected to a mechanical load through a gear reduction. The motor produces a torque in linear proportion to the current delivered from a constant-current source. The mechanical load is a disk with inertia J, and a translational spring and dashpot connected to the edge of the disk. Write the modeling equations for this system, and derive a transfer function for the angular position of the disk as a function of the input current.



Example 3

The ideal torque of a DC motor is linearly proportional to the current.

$$T_m = K_m\, i_m$$

The torque balance on the motor shaft considers the torque provided by the torque motor, the rotational inertia, and the torques due to the translational components.

$$T_m - J\,\dot{\omega}_m + \frac{d}{2}[F_x] = 0$$

The forces of the translational components at the edge of the disk are

$$F_x = -m\,\ddot{x} - b\,\dot{x} - k\,x$$

For small deflection angles, the translational motion can be related to the rotational motion.

$$x = \frac{d}{2}\,\theta$$

The torque balance equation can be restated with the above substitutions.

$$T_m = J\,D^2\theta + \frac{d}{2}\left[m\,D^2x + b\,Dx + k\,x\right] = J\,D^2\theta + \left(\frac{d}{2}\right)^2\left[m\,D^2\theta + b\,D\theta + k\,\theta\right]$$

Example 3

$$T_m = \left[J + \left(\frac{d}{2}\right)^2 m \right] D^2 \theta + b \left(\frac{d}{2}\right)^2 D\theta + k \left(\frac{d}{2}\right)^2 \theta = K_m\, i_m$$

With a current input, the transfer function for the angular position can be written as follows.

$$\theta = \frac{K_m\, i_m}{\left[J + \left(\frac{d}{2}\right)^2 m \right] D^2 + b \left(\frac{d}{2}\right)^2 D + k \left(\frac{d}{2}\right)^2}$$

Normalizing with respect to the lowest order coefficient yields

$$\theta = \frac{\dfrac{K_m}{k\,(d/2)^2}\, i_m}{\left[\dfrac{J}{k\,(d/2)^2} + \dfrac{m}{k} \right] D^2 + \dfrac{b}{k} D + 1}$$

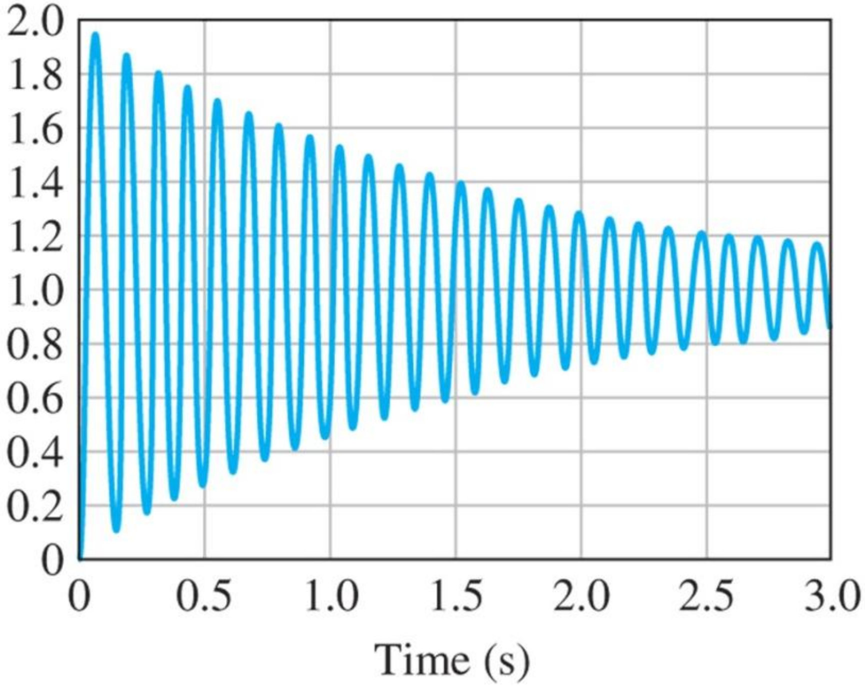
Pair-Share Exercise: Example 4

Simulate the step input of the system assuming zero initial conditions:

$$\frac{\theta}{i_m} = \frac{2700}{s^2 + 1.25s + 2701}$$

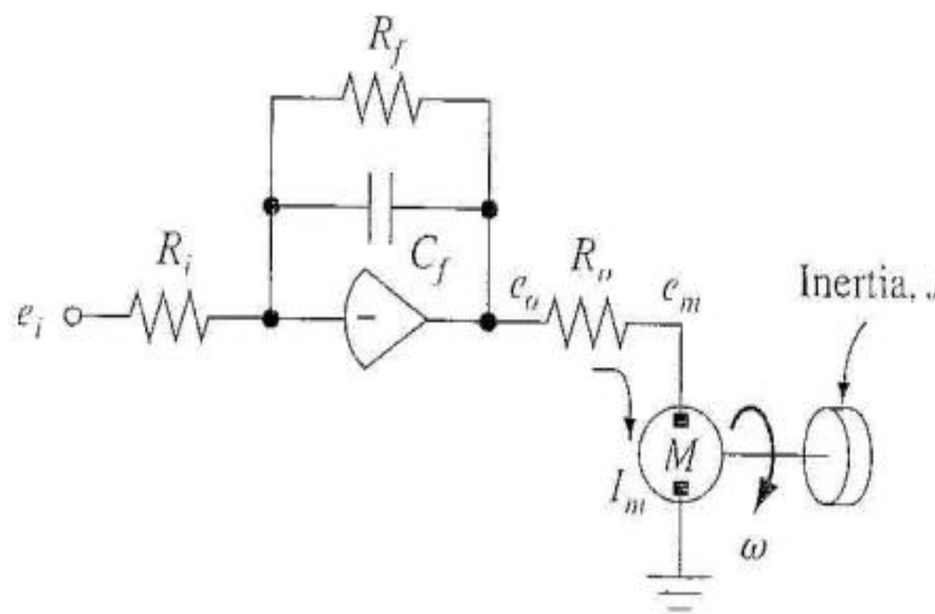
Matlab code:

```
num = [2700];
den=[1 1.25 2701];
sys=tf(num,den);
t=[0:.05:3];
[y,t]=step(sys,t);
plot(t,y); xlabel('Time (s)');
ylabel('position (rad)')
```



Pair-Share: Example 5

A small DC motor is used to spin a disk with an op-amp as shown. In this application, we do not want the disk to spin up very fast, so a dynamic filter is added to the op-amp. The op-amp acts as an ideal voltage source with an output impedance R_0. Write the modeling equation for this system, and derive a state-space representation of the system using the op-amp output e_0 and the shaft speed ω as state variables. Is this a stable system?



Example 5

The op amp can be treated as an ideal isolated component; thus, the model for the op amp can be stated as follows.

$$e_o = \frac{\dfrac{R_f \dfrac{1}{C\,D}}{R_f + \dfrac{1}{C\,D}}}{R_i} e_i = \frac{\dfrac{R_f}{C\,D\,R_f + 1}}{R_i} e_i = \frac{\dfrac{R_f}{R_i}}{R_f C\,D + 1} e_i$$

The output voltage of the op amp can be treated as an ideal voltage source, thus, the voltage to the motor is given by

$$e_m = e_o - R_o\,I_m$$

The model for the electric motor relates the torque and current, and the voltage and speed with a motor constant K_m.

$$T_m = K_m\,I_m$$

$$e_m = K_m\,\omega_m$$

A torque balance on the motor shaft yields the following.

Example 5

$$T_m - J\,\dot{\omega} = 0$$

By rearranging the above equation, and substituting all of the previous equations into it, we arrive at the following.

$$J\,D\omega = T_m = K_m\,I_m = K_m\,\frac{e_o - e_m}{R_o} = K_m\left[\frac{e_o}{R_o} - \frac{K_m\,\omega}{R_o}\right] = K_m\,\frac{e_o}{R_o} - \frac{K_m^2}{R_o}\,\omega$$

Since we are interested in deriving a state-space representation, we can write the above equation as follows.

$$D\omega = \frac{K_m}{J\,R_o}\,e_o - \frac{K_m^2}{J\,R_o}\,\omega$$

Restating the op amp equation yields the following.

$$\left[R_f C\,D + 1\right]e_o = \frac{R_f}{R_i}\,e_i \qquad \text{or} \qquad De_o = \frac{1}{R_i\,C}\,e_i - \frac{1}{R_f C}\,e_o$$

Example 5

A state-space representation of this system can be obtained by noting that the op amp equation gives us one state variable, and the torque equation above gives us another. The input to the system is the voltage input signal e_i and the state variables are the op amp output and the shaft speed.

$$u_1 = e_i$$

$$x_1 = e_o \qquad \dot{x}_1 = \frac{1}{R_i C} u_1 - \frac{1}{R_f C} x_1 \qquad \text{with} \qquad x_1(0) = e_o(0)$$

$$x_2 = \omega \qquad \dot{x}_2 = \frac{K_m}{J R_o} x_1 - \frac{K_m^2}{J R_o} x_2 \qquad \text{with} \qquad x_2(0) = \omega(0)$$

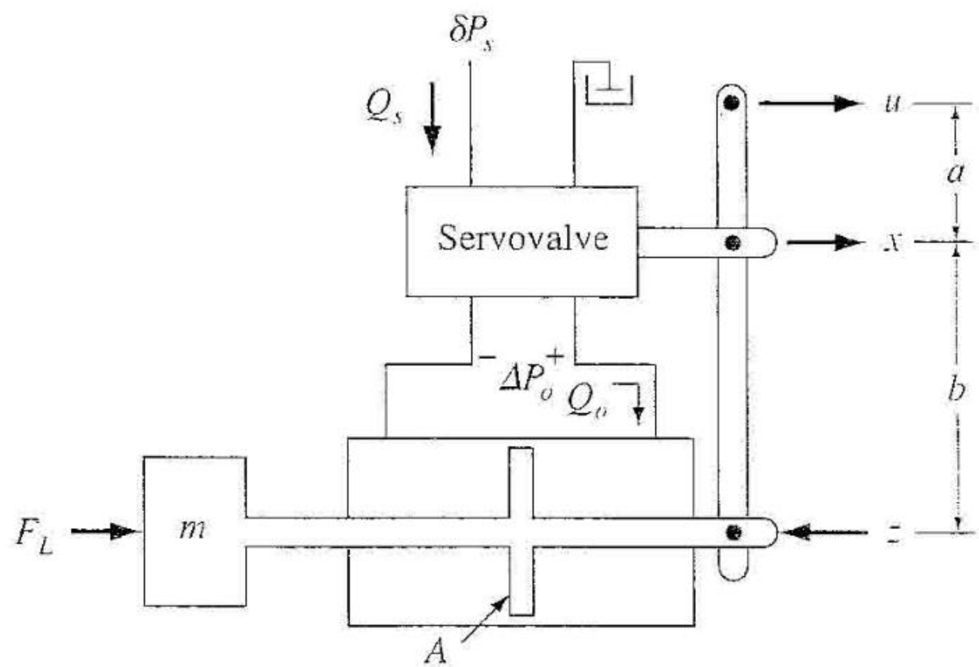
Fluid-Mechanical Systems

Fluid-Mechanical Systems

- Wide range of applications in precision motional control at high power levels
  - Flight controls (ailerons, rudder, stabilizer)
  - Automotive (power steering) and industrial
- Provide high power actuation in small volume, high responsiveness
- Configuration
  - Servo valve capable of controlling pressure and flow
  - Mechanical/Electrical feedback with actuator output

Hydraulic Position Servo

- Servo valve provides metered ΔP_0 and Q_0 in response to position x of spool valve
- Actuator moves M and experiences load F_L
- Mechanical level feedback: input u and actuator position z change, causing servo valve to modulate ΔP_0 and Q_0 to reduce error x of servo system



Hydraulic Position Servo

$Servovalve\,has\,linear\,characteristics\ for\,small$

$signal\,operations\,around\,null\ position\left(x = \Delta P_0 = Q_0 = 0\right)$

$$\Delta P_0 = G_p x - RQ_0$$

$Assume\,compressibility\,is\,small\left(i.e.bulk\,modulus\,is\,large\right),$

$areas\,equal\,on\,both\,sides\,of\ piston,\ flow\,enters\,one\,side\,equal$

$flows\,pushed\,on\,the\,other\,side,\ from\,continuity\,equation,$

$$Q_0 = \dot{V} = A\dot{z}$$

$Force\,balance,$

$$M\ddot{z} = A\Delta P_0 - F_L$$

$Kinematics\,of\ feedback\,lever,$

$$x = \frac{b}{a+b}u,\ x = \frac{-a}{a+b}z$$

$$let\,b' = \frac{b}{a+b},a' = \frac{a}{a+b}$$

$$x = b'u - a'z$$

Hydraulic Position Servo

Substitute into force balance equation and form TF,

$$z = \frac{\frac{b'}{a'}u - \frac{1}{k_s}F_L}{\frac{M}{k_s}D^2 + \frac{RA^2}{k_s}D + 1}$$

Static gain,

$$G_s = \left.\frac{\partial z}{\partial u}\right|_{D=0} = \frac{b'}{a'} = \frac{b}{a} \ (indep. of\ characteristics\ of\ other\ components)$$

$$Static\ stiffness\ k_s = -\left.\frac{\partial F_L}{\partial z}\right|_{D=0} = G_p A a'$$

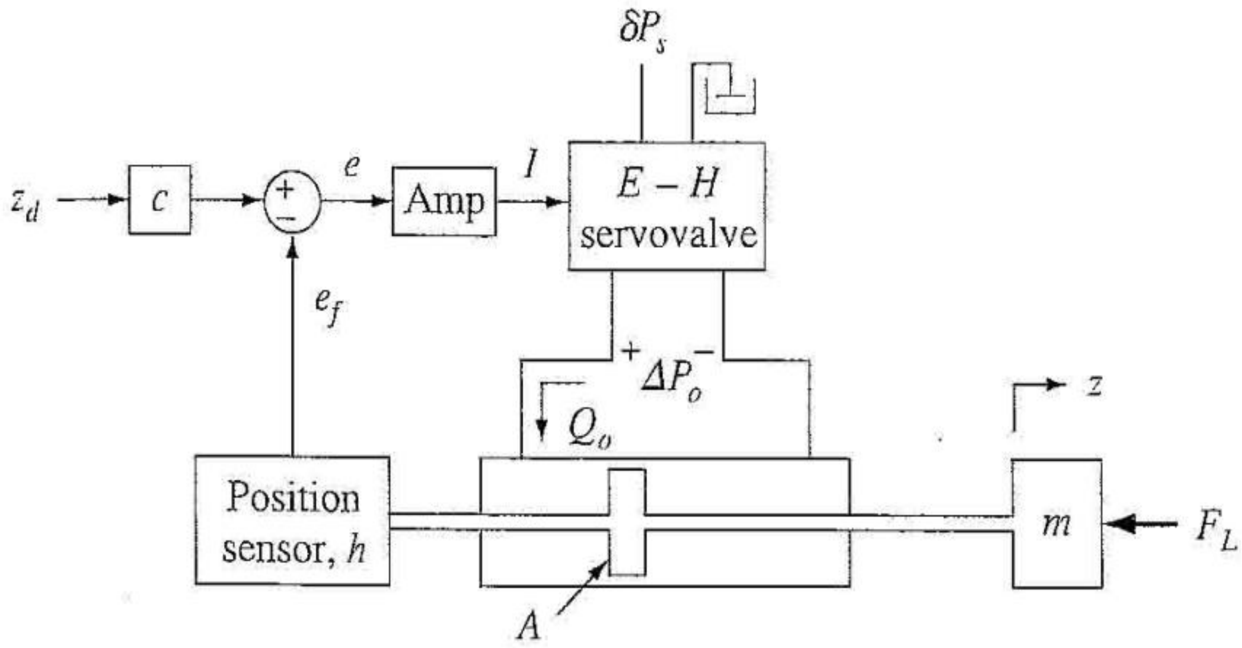
Second order system,

$$\omega_n = \sqrt{\frac{k_s}{M}}, \zeta = \frac{RA^2}{2\sqrt{k_s M}}$$

Sizing valve, actuator, feedback to get responsive, stiff servo

Electro-Hydraulic Position servo

Electro-Hydraulic Position Servo



Electro-hydraulic position servo control system

Electro-Hydraulic Position Servo

$$\Delta P_o = G_p I - R Q_o$$

$$Q_o = \dot{V} = A\dot{z}$$

$$M\ddot{z} + B\dot{z} = A\Delta P_o - F_L$$

$$E_f = hz$$

$$I = G_a\left(1 + \frac{D}{\omega_d}\right)(c z_d - E_f)$$

$$z = \frac{\left(\frac{D}{\omega_d} + 1\right)\frac{c}{h} z_d - \frac{1}{k_s} F_L}{\frac{M}{k_s} D^2 + \left(\frac{B + RA^2}{k_s} + \frac{1}{\omega_d}\right) D + 1}$$

$$k_s = G_a G_p A h$$

$$G_s = \frac{c}{h}$$

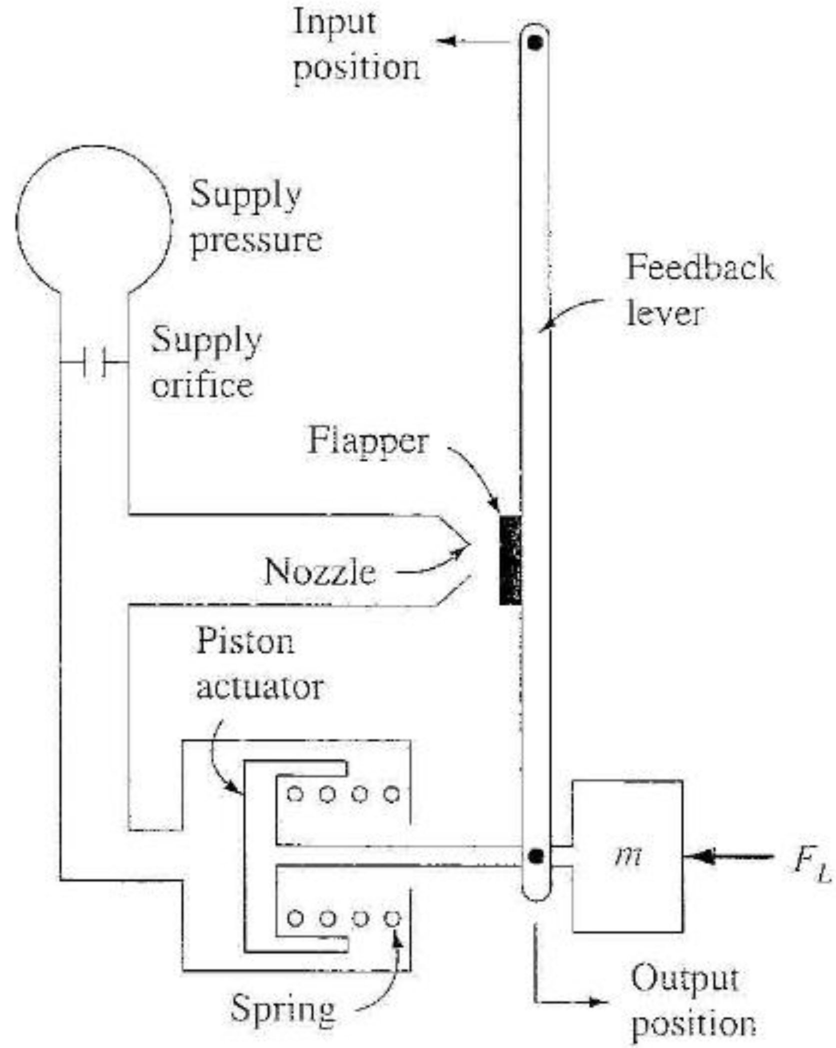
$$\omega_n = \sqrt{\frac{k_s}{M}}$$

$$\zeta = \frac{B + RA^2 + k_s / \omega_d}{2\sqrt{k_s M}}$$

Pneumatic Position Servo

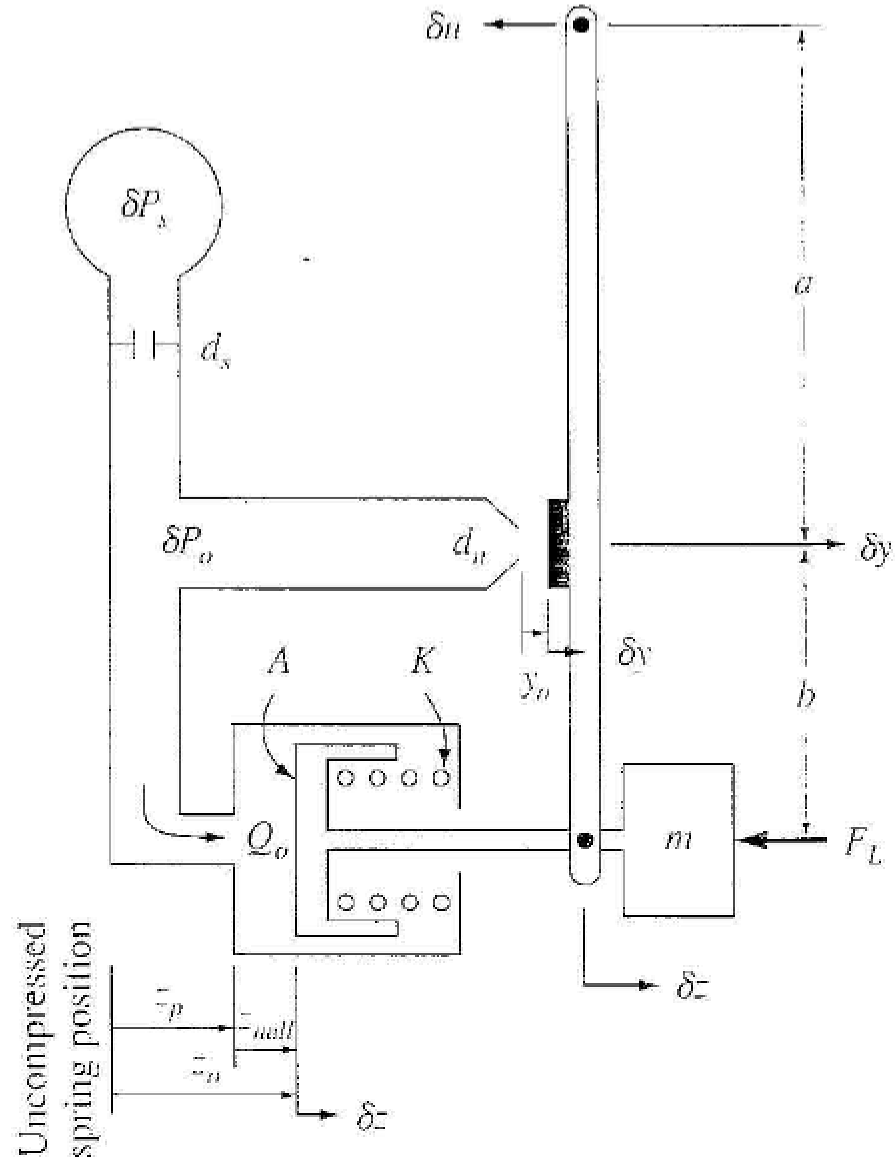
Pneumatic Position Servo

Servo made of three-way valve connected to spring-loaded actuator with mechanical lever feedback



Pneumatic Position Servo

- Decreasing y causes increase in δP_0 and output position δz
- If δu held constant, feedback lever causes valve to return to steady-state position
- If F_L causes decrease in z, valve will be actuated, pressure to actuator will increase a force to oppose F_L



Pneumatic Position Servo

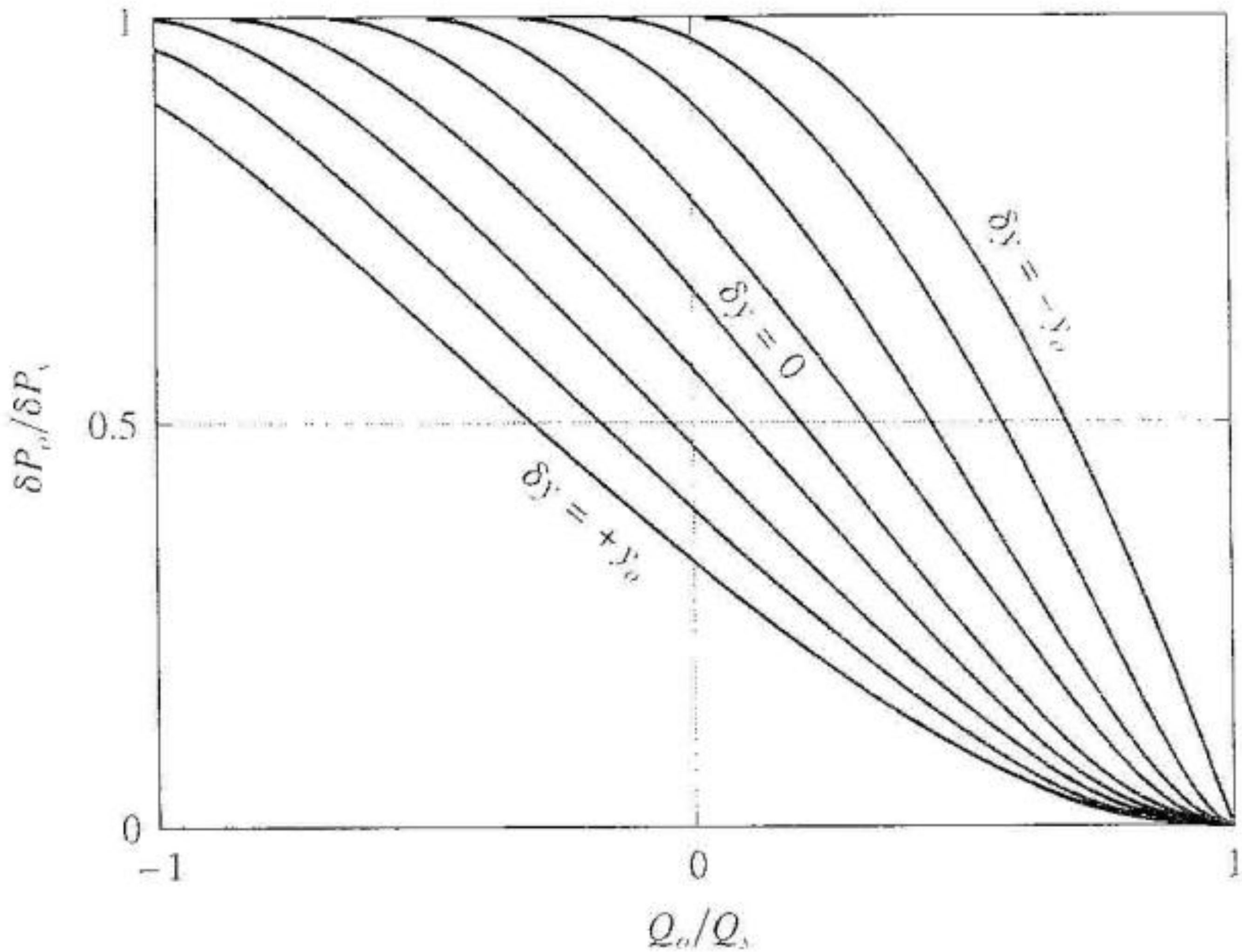
- In this system, actuator stroke starts at an unactuated position (z_0,y_0,u_0) when supply pressure is off and increases to null position when supply pressure is activated
- Examine small variations δz,δy,δz in positions around null operation:

  $z = z_0 + \delta z$

  $y = y_0 + \delta y$

  $u = u_0 + \delta u$
- Nonlinear system difficult to analyze, linearization to obtain transfer function

Pneumatic Position Servo



Nonlinear characteristics of three-way flapper-nozzle valve

Pneumatic Position Servo

Linear Analysis

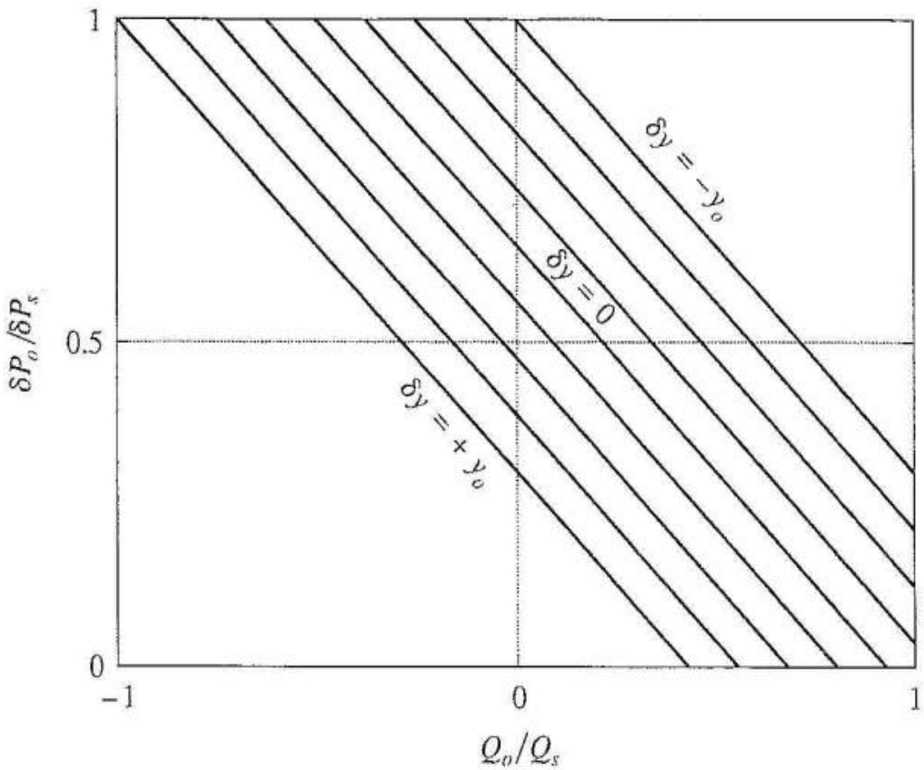
$$\delta P_o = \delta P_o^* - G_p \delta y - R_o Q_o$$

$$\delta P_o^* = \frac{\delta P_s}{\left[1+\alpha^{*2}\right]} = 0.671\,\delta P_s$$

$$G_p = \frac{\dfrac{\delta P_s}{y_o}}{\left[1+\dfrac{\alpha^{*2}+\alpha^{*-2}}{2}\right]} = 0.441\frac{\delta P_s}{y_o}$$

$$R_o = \frac{\dfrac{2\delta P_s}{Q_s^*}}{\alpha^* \sqrt{1+\alpha^{*2}}\left[1+\alpha^{*-2}\right]} = 0.770\frac{\delta P_s}{Q_s^*}$$

Pneumatic Position Servo



Linearized pressure-flow characteristics of three-way flapper-nozzle valve

Pneumatic Position Servo

$$Q_o = \frac{V}{\beta}\delta\dot{P}_o + \dot{V}$$

$$V \approx V_o + Az_{null} = V^*$$

$$\dot{V} = A\delta\dot{z}$$

$$\beta = n(\delta P_o + P_{atm}) \approx n(\delta P_o^* + P_{atm}) = \beta^*$$

$$M\delta\ddot{z} + k(z_p + z_{null} + \delta z) = A\delta P_o - F_L$$

$$\delta_y = a'\delta z - (1 - a')\delta u$$

$$\delta P_o = \frac{\delta P_o^* - G_p\delta y - R_o A\delta\dot{z}}{R_o \dfrac{V^*}{\beta^*} D + 1}$$

Pneumatic Position Servo

$$\delta z = \frac{G_s \delta u + \delta z_{ss} - \dfrac{\left(R_o \dfrac{V^*}{\beta^*} D + 1 \right)}{k_s} F_L}{R_o \dfrac{V^* M}{\beta^* k_s} D^3 + \dfrac{M}{k_s} D^2 + \dfrac{R_o A^2}{k_s} \left(1 + \dfrac{V^* / \beta^*}{A^2 / k} \right) D + 1}$$

$$k_s = k + G_p A a'$$

$$\delta z_{ss} = \frac{A \delta P_o^* - k(z_p + z_{null})}{k_s} = 0$$

$$G_s = \frac{(1 - a') / a'}{\left(1 + \dfrac{k}{G_p A a'} \right)}$$

Example 7

In what follows, use the basic models for flows in the flapper-nozzle valve to derive the normalized model for the flapper-nozzle valve. Plot the output pressure δP_0 versus the normalized valve clearance α with no output flow. Show that the point $\alpha = 0.7$ is a good trade-off from among the maximum gain, the minimum mean output pressure, and the maximum linear modulation of the output pressure. Show that the null output pressure is $0.671\ \delta P_s$. Linearize this model. The parameters for flow in the flapper-nozzle valve are:

Example 7

$$Supply\ \ flow: \qquad Q_s = C_{ds}A_s\sqrt{\frac{2(\delta P_s - \delta P_0)}{P}}$$

$$Maximum\,supply\ \ flow: \quad Q_s^* = C_{ds}A_s\sqrt{\frac{2\delta P_s}{P}}$$

$$Output\ \ flow: \qquad Q_0 = Q_s - Q_n$$

$$Nozzle\ \ flow: \qquad Q_n = C_{dn}A_n\sqrt{\frac{2\delta P_0}{P}}$$

$$Supply\,orifice\ \ flow\,area: A_s = \frac{\pi}{4}d_s^2$$

$$Nozzle\ \ circumferential\ \ flow\,area: A_n = \pi d_n y$$

Example 7

In your derivation of the output flow normalized to the maximum supply flow, you should find a natural grouping of terms such as the following:

$$Normalized\ valvestroke:\ \alpha = 4\frac{C_{dn}}{C_{cs}}(\frac{d_n}{d_s})^2\frac{y}{d_n}$$

For symmetric operation, the null position of the valve is y_0, and the valve can stroke $\pm\delta y$

Example 7

$$Valve\ \ position: \qquad y = y_0 + \delta y$$

$$Thus,\ the\ null\ \ value\ of\ \alpha: \qquad \alpha^* = 4\frac{C_{dn}}{C_{cs}}(\frac{d_n}{d_s})^2\frac{y_0}{d_n}$$

For this model to be correct, the circumferential (or curtain) flow area A_n of the nozzle should be smaller than the area A_s of the supply orifice; that is,

$$\frac{y_0}{d_n}\langle 0.125$$

Example 7: Solution

The net output flow can be stated as a function of the supply flow and the leakage flow out the nozzle.

$$Q_o = Q_s - Q_n$$

The supply flow is

$$Q_s = C_{ds} A_s \sqrt{\frac{2(\delta P_s - \delta P_o)}{\rho}} \qquad \text{where} \qquad A_s = \frac{\pi}{4} d_s^2$$

The leakage flow out the nozzle is

Example 7: Solution

$$Q_n = C_{dn} A_n \sqrt{\frac{2\,\delta P_o}{\rho}} \qquad \text{where} \qquad A_n = \pi d_n y$$

In order to help the normalization, we can define the maximum possible supply flow (when δP_o is equal to zero).

$$Q_s^* = C_{ds} A_s \sqrt{\frac{2\,\delta P_s}{\rho}}$$

Substituting these flow equations into the first equation for the output flow and normalizing by the maximum flow yields the following.

$$\frac{Q_o}{Q_s^*} = \frac{Q_s}{Q_s^*} - \frac{Q_n}{Q_s^*} \qquad \text{or} \qquad \frac{Q_o}{Q_s^*} = \sqrt{\frac{(\delta P_s - \delta P_o)}{\delta P_s}} - \frac{C_{dn} A_n}{C_s A_s}\sqrt{\frac{\delta P_o}{\delta P_s}}$$

Using the area equations yields

$$\frac{Q_o}{Q_s^*} = \sqrt{1 - \frac{\delta P_o}{\delta P_s}} - \frac{4\,C_{dn}\,\pi d_n y}{C_{ds}\,\pi d_s^2}\sqrt{\frac{\delta P_o}{\delta P_s}}$$

Example 7: Solution

It is convenient to define the grouping of terms in the above equation as the normalized valve stroke α.

$$\alpha = 4\frac{C_{dn}}{C_{ds}}\frac{d_n\,y}{d_s^2} = 4\frac{C_{dn}}{C_{ds}}\left(\frac{d_n}{d_s}\right)^2\frac{y}{d_n}$$

If we consider that the flapper is normally at some null position y_0 and that flapper motion is defined as some δy from that null position, we can arrive at the following.

$$y = y_0 + \delta y$$

then $\alpha = \alpha^* + 4\frac{C_{dn}}{C_{ds}}\left(\frac{d_n}{d_s}\right)^2\frac{\delta y}{d_n}$ where $\alpha^* = 4\frac{C_{dn}}{C_{ds}}\left(\frac{d_n}{d_s}\right)^2\frac{y_0}{d_n}$

Thus, the full nonlinear model can be stated as follows.

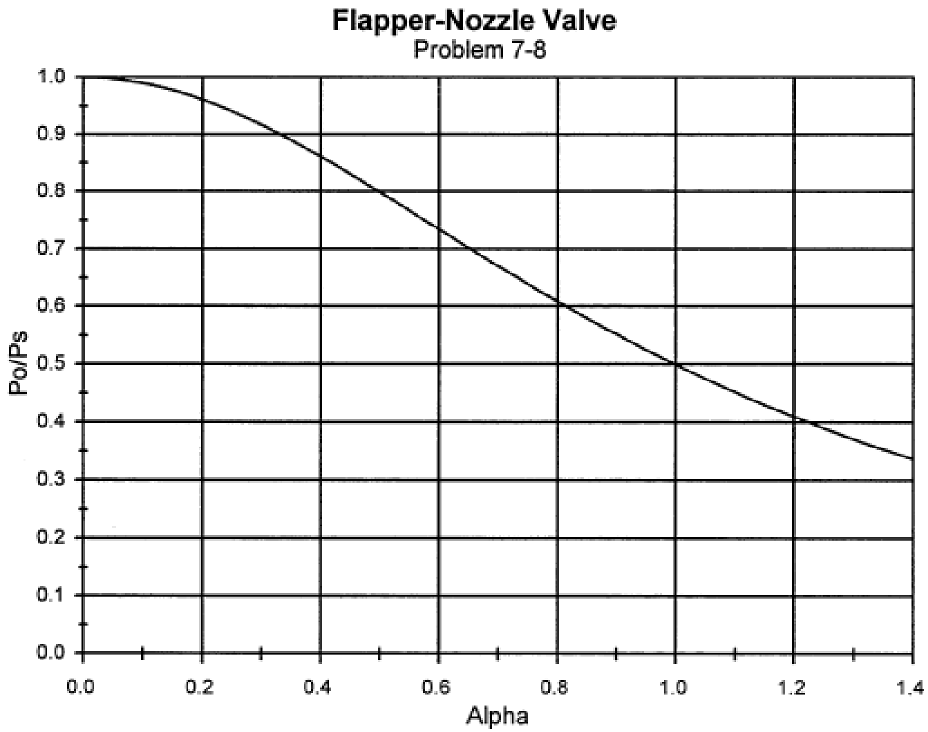
$$\frac{Q_o}{Q_s^*} = \sqrt{1-\frac{\delta P_o}{\delta P_s}} - \left(\alpha^* + 4\frac{C_{dn}}{C_{ds}}\left(\frac{d_n}{d_s}\right)^2\frac{\delta y}{d_n}\right)\sqrt{\frac{\delta P_o}{\delta P_s}}$$

Example 7: Solution

The next thing that we need to do is to determine the optimum value for α^*. What we have to do is set the flow to zero and resolve for the pressure as a function of α.

$$\frac{\delta P_o}{\delta P_s} = \frac{1}{1+\left(\alpha^* + \delta\alpha\right)^2} = \frac{1}{1+\alpha^2}$$

This pressure response function is graphed below.



Example 7: Solution

From this pressure function, we can determine the gain, the mean pressure, and the pressure range. The gain is the slope of the pressure versus alpha. The slope is negative, so we will acknowledge this fact by a minus sign.

$$\text{gain} = \frac{\partial \frac{\delta P_o}{\delta P_s}}{\partial \alpha} = \frac{-2\,\alpha}{\left(1+\alpha^2\right)^2}$$

Since our motion will be defined as a variation $\delta\alpha$ from some null position α^*, we need to find the optimum null position with $\pm\delta\alpha$ from that null position. Thus, in the following, we will find the effect of the characteristics as a function of the null position.

$$-\text{gain} = \frac{2\,\alpha^*}{\left(1+\alpha^{*2}\right)^2}$$

Example 7: Solution

The mean pressure represents the average of the pressures with the maximum and the minimum input strokes $\pm\delta\alpha$. It is desirable to keep the mean pressure low.

$$\text{mean} = \frac{\left.\dfrac{\delta P_o}{\delta P_s}\right|_{\alpha^*-\delta\alpha} + \left.\dfrac{\delta P_o}{\delta P_s}\right|_{\alpha^*+\delta\alpha}}{2} = \frac{1+\dfrac{1}{1+\left(2\alpha^*\right)^2}}{2}$$

The range of pressure from the maximum output to the minimum output represents the largest pressure signal that could be generated. We want the range to be as large as possible.

$$\text{range} = \left.\frac{\delta P_o}{\delta P_s}\right|_{\alpha^*-\delta\alpha} - \left.\frac{\delta P_o}{\delta P_s}\right|_{\alpha^*+\delta\alpha} = 1-\frac{1}{1+\left(2\alpha^*\right)^2}$$

Example 7: Solution

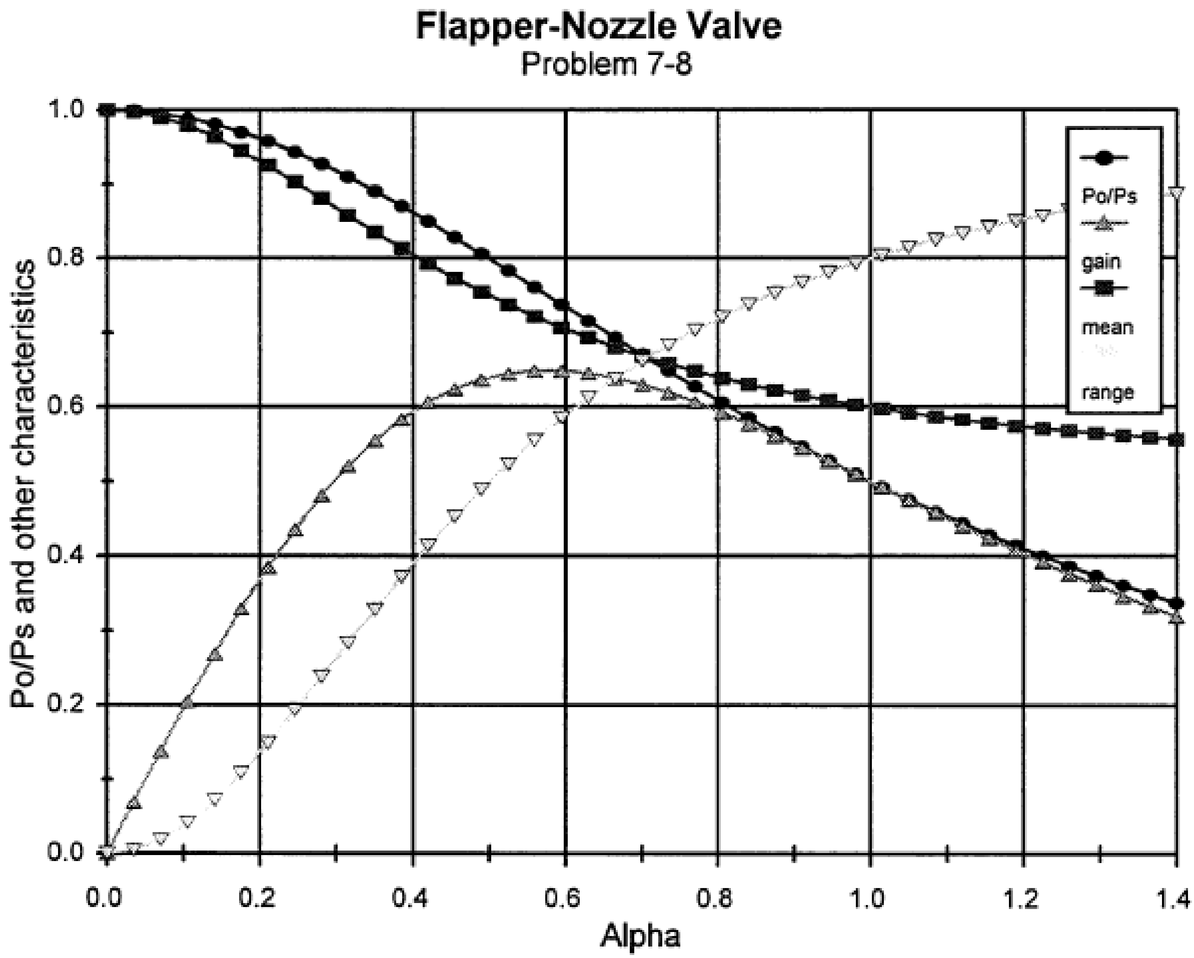
All of these performance factors are plotted below. From this graph, we can see that the gain reaches a maximum around $\alpha^* = 0.6$. Actually, we can mathematically find the optimum of the gain function to be $\sqrt{\frac{1}{3}} = 0.577$. The mean value starts at 1 (with $\alpha^* = 0$) and decreases; but after $\alpha^* = 0.8$, the decrease is small. Therefore, we are motivated to have α^* larger. The range of pressure increases from zero (with $\alpha^* = 0$), and starts to flatten out with large α^*; therefore, our motivation is to make α^* large again.

With these considerations in mind, we select $\alpha^* = 0.7$ as a good trade-off between maximum gain, maximum range, and minimum mean pressure.

With $\alpha^* = 0.7$, the output pressure at null can be found to be

$$\frac{\delta P_o}{\delta P_s} = \frac{1}{1+\alpha^2} = \frac{1}{1+0.7^2} = 0.671$$

Example 7: Solution



Example 7: Solution

The next task is to linearize the complete equation for the pressure and flow as a function of alpha given previously.

$$\frac{Q_o}{Q_s^*} = \sqrt{1-\frac{\delta P_o}{\delta P_s}} - \left(\alpha^* + 4\frac{C_{dn}}{C_{ds}}\left(\frac{d_n}{d_s}\right)^2 \frac{\delta y}{d_n}\right)\sqrt{\frac{\delta P_o}{\delta P_s}}$$

We can use symbols for the normalized flow and pressure variables and simplify to the following.

$$\overline{Q} = \sqrt{1-\overline{P}} - \alpha\sqrt{\overline{P}}$$

Now, we are hunting for a linearized model of this equation around the null operating point of $\alpha = \alpha^* = 0.7$ (thus $\delta\alpha = 0$) and $\overline{P} = \overline{P}^* = 0.671$.

$$\overline{Q} \approx \left.\frac{\partial\overline{Q}}{\partial\alpha}\right|_{null}\left(\alpha-\alpha^*\right) + \left.\frac{\partial\overline{Q}}{\partial\overline{P}}\right|_{null}\left(\overline{P}-\overline{P}^*\right)$$

$$\overline{Q} \approx -\sqrt{\overline{P}^*}\left(\alpha-\alpha^*\right) + \left[\frac{-1}{2\sqrt{1-\overline{P}^*}} - \frac{\alpha^*}{2\sqrt{\overline{P}^*}}\right]\left(\overline{P}-\overline{P}^*\right)$$

if we use $\alpha - \alpha^* = \delta\alpha$

$$-\overline{Q} \approx \sqrt{\overline{P}^*}\,\delta\alpha + \left[\frac{1}{2\sqrt{1-\overline{P}^*}} + \frac{\alpha^*}{2\sqrt{\overline{P}^*}}\right]\left(\overline{P}-\overline{P}^*\right)$$

Example 7: Solution

Rearranging this equation for P as a function of Q and α (note that P^* and α^* are constants)

$$\overline{P} = \overline{P}^* - \frac{1}{\left[1 + \frac{\alpha^{*2} + \alpha^{*-2}}{2}\right]} \frac{\delta\alpha}{\alpha^*} - \frac{2}{\alpha^* \sqrt{1+\alpha^{*2}}\left[1+\alpha^{*-2}\right]} \overline{Q}$$

If we reuse the ratio notation and unnormalize, we can get the following.

$$\delta P_o = \delta P_o^* - \frac{\delta P_s}{\left[1 + \frac{\alpha^{*2} + \alpha^{*-2}}{2}\right]} \frac{\delta\alpha}{\alpha^*} - \frac{2\frac{\delta P_s}{Q_s}}{\alpha^* \sqrt{1+\alpha^{*2}}\left[1+\alpha^{*-2}\right]} Q_o$$

Since we are interested in the actual flapper motion, and by noting that $\frac{\delta\alpha}{\alpha^*} = \frac{\delta y}{y_0}$, we can state the final result.

$$\delta P_o = \delta P_o^* - G_p\, \delta y - R_o\, Q_o$$

Example 7: Solution

where the null output pressure is
$$\delta P_o = \frac{\delta P_s}{1+\alpha^{*2}} = 0.671\,\delta P_s$$

the static pressure gain is
$$G_p = \frac{\dfrac{\delta P_s}{y_0}}{\left[1+\dfrac{\alpha^{*2}+\alpha^{*-2}}{2}\right]} = 0.441\,\frac{\delta P_s}{y_0}$$

and the output impedance is
$$R_o = \frac{2\,\dfrac{\delta P_s}{Q_s}}{\alpha^{*}\sqrt{1+\alpha^{*2}}\left[1+\alpha^{*-2}\right]} = 0.770\,\frac{\delta P_s}{Q_s}$$