Orifice Flow Resistance

Apply Bernoulli's equation for streamlines between 1 and 2, 2 and 3,

$$\frac{P_1}{\rho}+\frac{v_1^2}{2}=\frac{P_2}{\rho}+\frac{v_2^2}{2}=\frac{P_3}{\rho}+\frac{v_3^2}{2}$$

Equating volume flow rate at all cross sections,

$$Q=A_1v_1=A_2v_2=A_3v_3=A_4v_4$$

Assume sudden expansion between 2 and 4, then there is a velocity head loss and pressures are same at 3 and 4, hence $P_3=P_4$ (easy to measure)

Assume $A_1>A_2$ and $A_1>A_3$, then $v_1<v_3$ and $v_4 < v_3$, so v_1 and v_4 are negligible

Combine above equations with assumptions,

$$P_1-P_4=\frac{\rho}{2}v_3^2$$

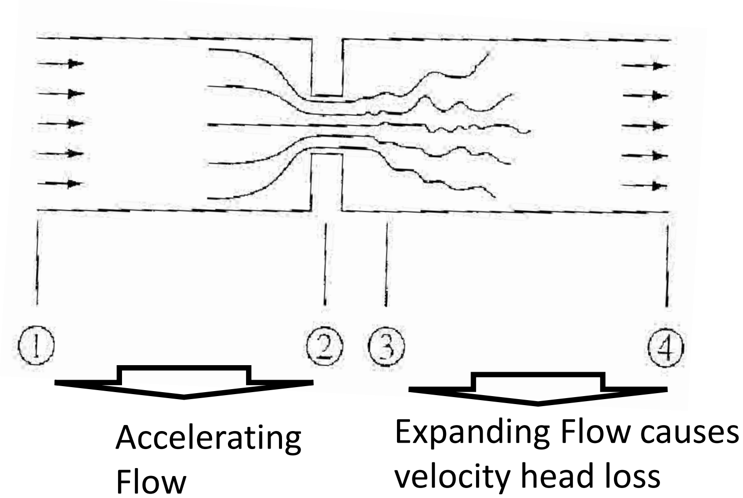
Use discharge coefficient C_d (combined effects of losses betwen 1 and 3, and representing A_3 in term of A_2)

$$\delta P=P_1-P_4=\frac{\rho}{2C_d^2A_2^2}Q^2 \qquad (orifice\ \ flow\ \ relation)$$

$$or\ Q=C_dA_2\sqrt{\frac{2\delta P}{\rho}}=\sqrt{\frac{|\delta P|}{K}}sign(\delta P)$$

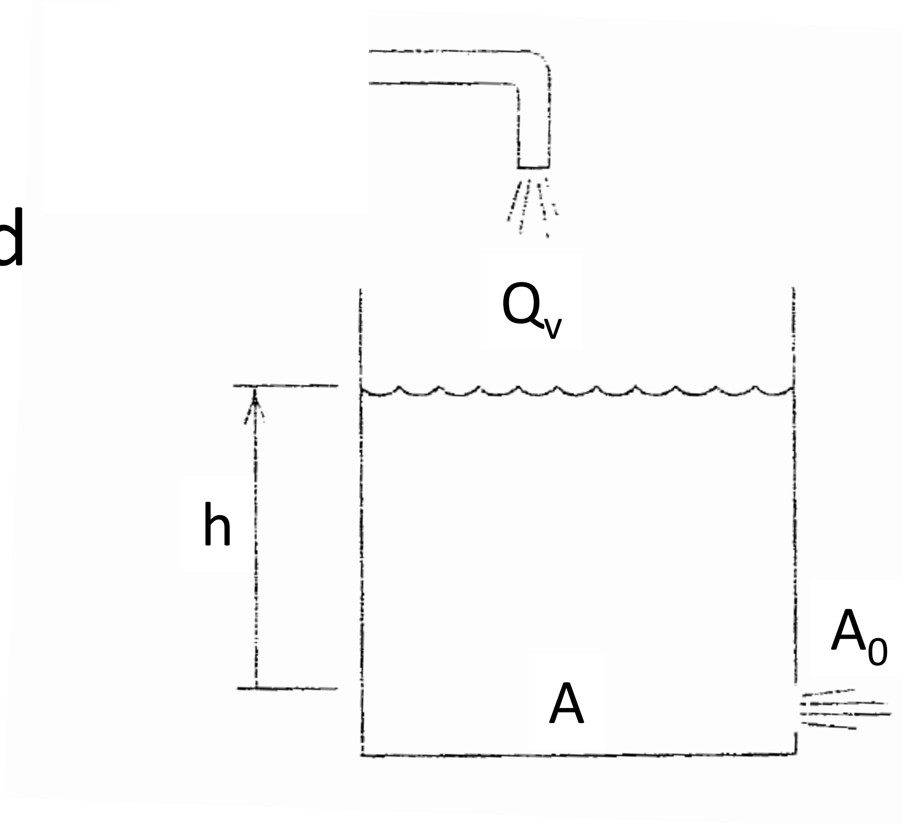
$$where\ \ K=\frac{\rho}{2C_d^2A_2^2};\ sign(x)=1\ if\ \ x>0;\ sign(x)=-1\ if\ \ x<0$$

$$C_d=C_d\,(geometry, roughness, orifice\ configuration, N_r)\sim .5\ to\ 1$$



Example 5: Tank with an Orifice

- The tank shown has an orifice in its side wall. The orifice area is A_0 and the bottom area of the tank is A. The liquid height above the orifice h. The volume inflow rate is Q_v. Develop a model of the height h with Q_v as the input.



Example 5: Tank with an Orifice

From conservation of mass and the orifice flow relation,

$$\dot{m}_{\text{tank}} = \dot{m}_{in} - \dot{m}_{out}$$

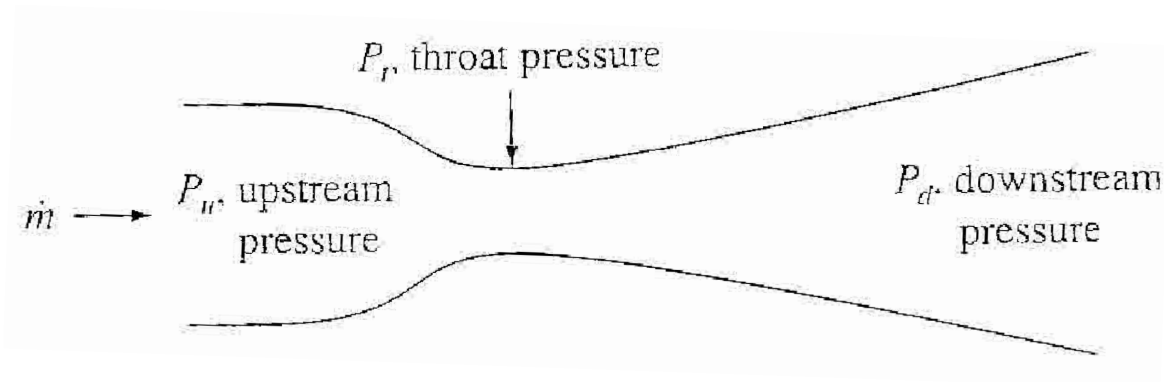
$$\rho A \frac{dh}{dt} = \rho Q_v - \rho C_d A_0 \sqrt{\frac{2\delta P}{\rho}}$$

$$but\ \ \delta P = \rho g h,$$

$$\rho A \frac{dh}{dt} = \rho Q_v - \rho C_d A_0 \sqrt{2gh}$$

This is a nonlinear equation, how can it be analyzed?

Compressible Flow Resistance



- Similar to orifice flow, but considers density variation and has higher degree of nonlinearity and complex equations
- For compressible flow, mach number, $N_m = v/c_0$,
  - cannot exceed 1 at the throat
  - equal 1 for air at $P_t/P_u = .528$ (flow is choked)
- Once choked, N_m at throat stays at 1-> $Q = A_t c_0$ = constant
- Due to compressibility, density can vary -> mass flow rate ($=\rho_t Q$) can vary if upstream conditions change

Compressible Flow Resistance

The critical pressure ratio determines whether flow is choked,

$$P_{cr} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}$$

Mass flow rate per unit area,

$$\frac{\dot{m}}{C_d A} = \sqrt{\frac{2}{RT^*}}\sqrt{\frac{T^*}{T_u}}\sqrt{\frac{k}{k-1}}P_u\left[\left(\frac{P_t}{P_u}\right)^{\frac{2}{k}} - \left(\frac{P_t}{P_u}\right)^{\frac{k+1}{k}}\right]^{\frac{1}{2}} \quad (unchoked\ flow, \frac{P_t}{P_u} > P_{cr})$$

$$\frac{\dot{m}}{C_d A} = \sqrt{\frac{2}{RT^*}}\sqrt{\frac{T^*}{T_u}}\sqrt{\frac{k}{k-1}}P_u\left[\left(P_{cr}\right)^{\frac{2}{k}} - \left(P_{cr}\right)^{\frac{k+1}{k}}\right]^{\frac{1}{2}} \quad (choked\ flow, \frac{P_t}{P_u} = P_{cr})$$

where $T^* =$ *reference temperature*

P_t is not known usually. For orifice exits into large chamber, assume $P_t = P_d$.
One disadvantage of compressible equations: ability to solve for mass flow rate in terms of pressures, but not reverse.

Compressible Flow Resistance

- Approximation to compressible flow equation is possible by using incompressible flow equation $\delta P = \rho Q^2 / (2 C_d^2 A^2)$ and gas law, and substituting throat pressure with upstream pressure.

$$\frac{\dot{m}}{C_d A} = \sqrt{\frac{2}{RT^*}} \sqrt{\frac{T^*}{T_u}} P_u \left[\left(\frac{P_t}{P_u} \right) - \left(\frac{P_t}{P_u} \right)^2 \right]^{\frac{1}{2}} \quad (unchoked\ incompressible\ approx)$$

$$\frac{\dot{m}}{C_d A} = \sqrt{\frac{2}{RT^*}} \sqrt{\frac{T^*}{T_u}} \sqrt{\frac{k}{k-1}} P_u \left[(P_{cr}) - (P_{cr})^2 \right]^{\frac{1}{2}} \quad (choked\ incompressible\ approx)$$

- Can solve for P_t or P_u as a function of mass flow rate

Case Study:
Spring-Loaded Diaphragm Actuator

Spring-Loaded Diaphragm Actuator

- Read handout
- Work in groups to answer parts a, b, and c

Spring-Loaded Diaphragm Actuator

Model of system using incompressible orifice equation and no mass:

$$\delta P_0 - \delta P_1 = K_o\, Q^2 \,\mathrm{sign}(Q)\,, \quad K_o = \frac{\rho}{2\, C_d^2\, A_o^2}\,, \quad A\,\delta P_1 - k_s\, z = 0$$

$$Q = \frac{V}{\beta}\,\delta \dot{P}_1 + \dot{V}\,, \quad V = V_o + A\,z\,, \quad \dot{V} = A\,\dot{z}\,, \quad \beta = n\left(\delta P_1 + P_{atm}\right)$$

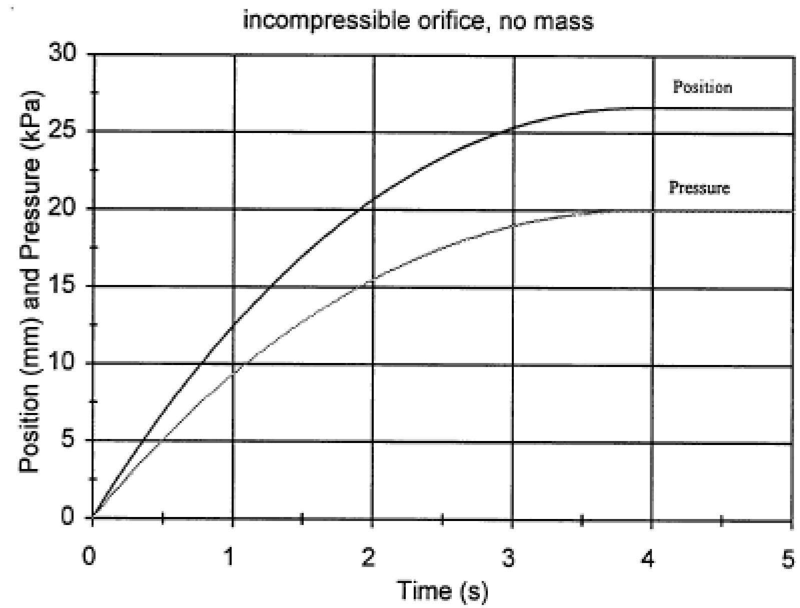
$$\dot{z} = \frac{\sqrt{\dfrac{\left|\delta P_0 - \dfrac{k_s}{A} z\right|}{K_o\, A^2}}\ \mathrm{sign}(\delta P_0 - \dfrac{k_s}{A} z)}{\left[\dfrac{V}{\beta}\dfrac{k_s}{A^2} + 1\right]}$$

state variables: $u_1 = \delta P_0\,, \quad x_1 = z$

Spring-Loaded Diaphragm Actuator

$$\dot{x}_1 = \frac{\sqrt{\frac{\left|u_1 - \frac{k_s}{A}x_1\right|}{K_o\,A^2}}\ \mathrm{sign}(u_1 - \frac{k_s}{A}x_1)}{\left[\frac{(V_o + A\,x_1)}{n\left(\frac{k_s}{A}x_1 + P_{atm}\right)}\frac{k_s}{A^2} + 1\right]}$$

outputs: $y_1 = z = x_1\ ,\quad y_2 = \delta P_1 = \frac{k_s}{A}x_1$



Spring-Loaded Diaphragm Actuator

Model of system using incompressible orifice equation with mass:

Change force balance to $A\,\delta P_1 - M\,\ddot{z} - k_s\,z = 0$:

$$\left[M\,D^2 + k_s\right]z = A\,\delta P_1\,, \quad \frac{V}{\beta}\,\delta\dot{P}_1 = \sqrt{\frac{\left|\delta P_0 - \delta P_1\right|}{K_o}}\,\mathrm{sign}(\delta P_0 - \delta P) - A\,\dot{z}$$

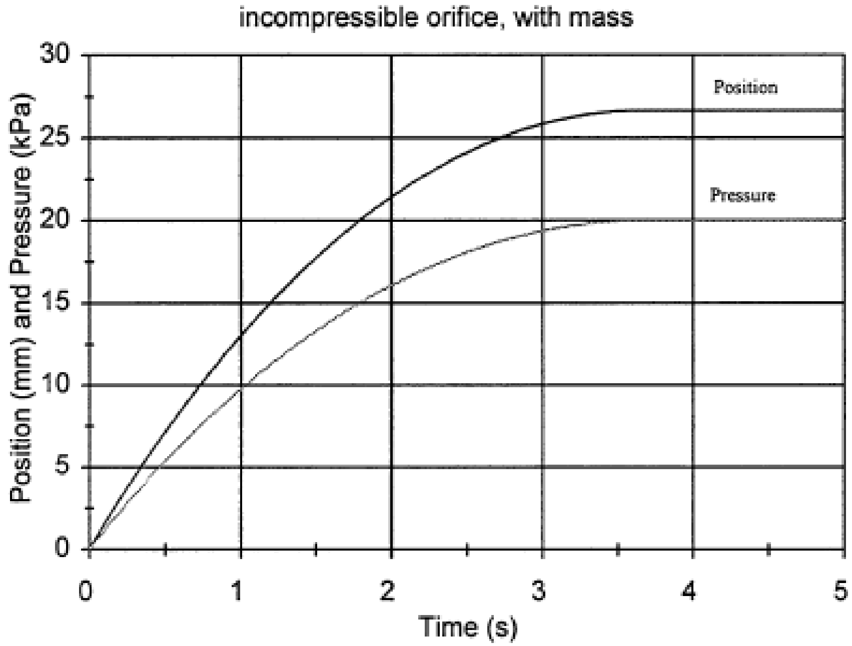
state variables: $u_1 = \delta P_0\,, \quad x_1 = z\,, \quad x_1 = \dot{z}\,, \quad x_1 = \delta P_1$

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = -\frac{k_s}{M}x_1 + \frac{A}{M}x_2$$

$$\dot{x}_3 = \frac{n\left(x_3 + P_{atm}\right)}{\left(V_o + A\,x_1\right)}\left\{\sqrt{\frac{\left|u_1 - x_3\right|}{K_o}}\,\mathrm{sign}(u_1 - x_3) - A\,x_2\right\}$$

outputs: $y_1 = z = x_1\,, \quad y_2 = \delta P_1 = x_3$

Spring-Loaded Diaphragm Actuator



Spring-Loaded Diaphragm Actuator

Model of system using compressible orifice equation (incompressible approximation) with mass:

Change flow equation to (neglecting temperature variations):

$$\dot{m}_{in} = C_d\, A_o \sqrt{\frac{2}{R\, T^*}}\, (\delta P_0 + P_{atm}) \sqrt{\left| P_r - P_r^2 \right|}\, \mathrm{sign}(P_r - P_r^2)\ , \quad C_d\, A_o = \sqrt{\frac{\rho_o}{2\, K_o}}$$

where $P_r = \dfrac{\delta P_1 + P_{atm}}{\delta P_0 + P_{atm}}$ and if $P_r \le 0.528$, then $P_r = 0.528$

Change continuity equation to:

$$\delta \dot{P}_1 = \frac{\beta}{V}\left[\frac{\dot{m}}{\rho_1} - A\, \dot{z} \right], \quad \rho_1 = \frac{\delta P_1}{R\, T_1}$$

state variables: $u_1 = \delta P_0$, $x_1 = z$, $x_1 = \dot{z}$, $x_1 = \delta P_1$

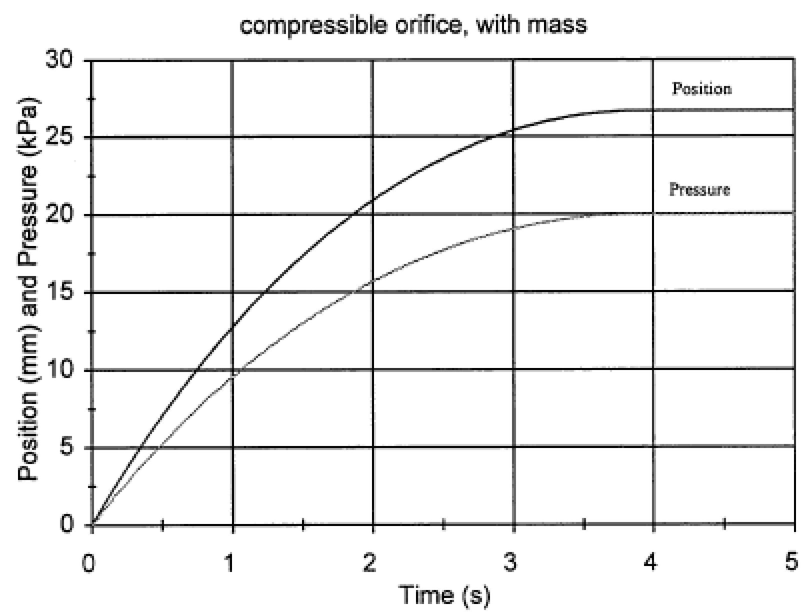
Spring-Loaded Diaphragm Actuator

$$\dot{x}_1 = x_2 \,, \quad \dot{x}_2 = -\frac{k_s}{M} x_1 + \frac{A}{M} x_2$$

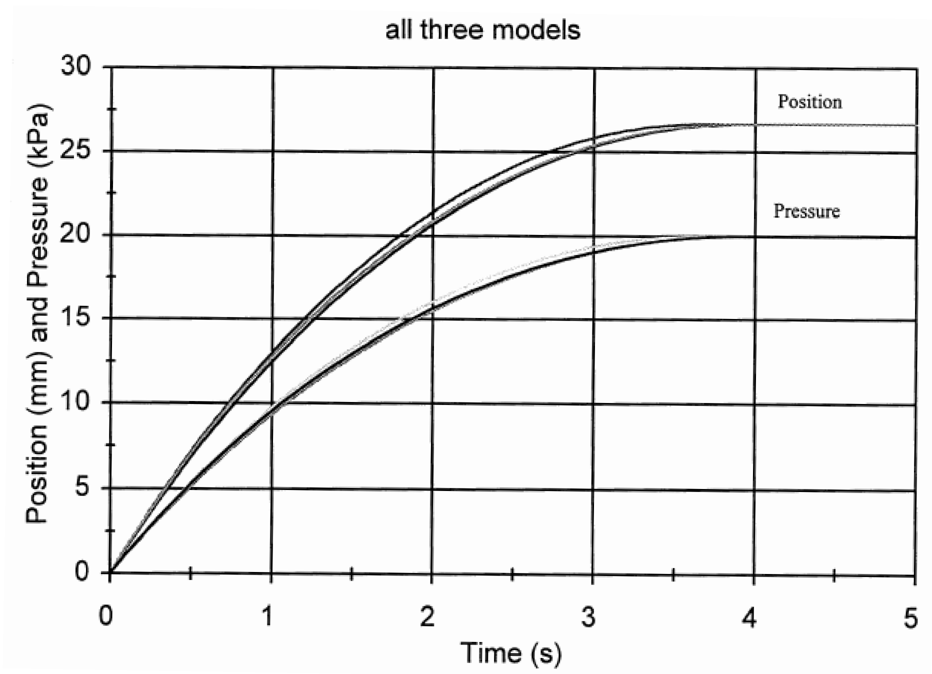
$$P_r = \frac{x_3 + P_{atm}}{u_1 + P_{atm}} \text{ and if } P_r \leq 0.528 \,, \text{ then } P_r = 0.528$$

$$\dot{x}_3 = \frac{n(x_3 + P_{atm})}{(V_o + A\,x_1)} \left\{ C_d\, A_o \sqrt{2\,R\,T_1}\, \frac{(\delta P_0 + P_{atm})}{(\delta P_1 + P_{atm})} \sqrt{\left| P_r - P_r^2 \right|}\, \text{sign}(P_r - P_r^2) - A\,x_2 \right\}$$

outputs: $y_1 = z = x_1 \,, \quad y_2 = \delta P_1 = x_3$

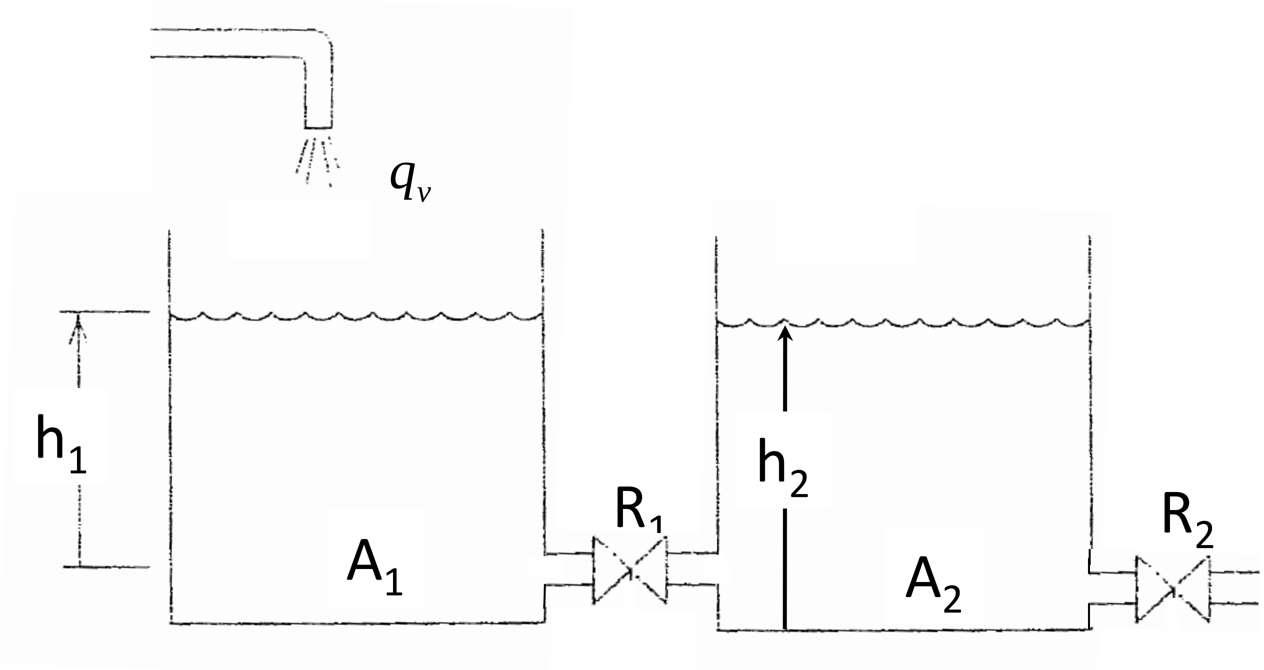


Spring-Loaded Diaphragm Actuator



Homework 5: Chapter 5

- 5.5, 5.11, 5.13, 5.15
- Two Interconnected Tanks Problem: Develop a model for the heights h_1 and h_2 in the liquid system shown in Figure below. The input volume flow rate q_v is given. Assume that laminar flow exists in the pipes. The laminar resistances are R_1 and R_2, and the bottom areas of the tanks are A_1 and A_2.



References

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- Close, C. M., Frederick, D. H., Newell, J. C., Modeling and Analysis of Dynamic Systems, Third Edition, Wiley, 2002
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