Modeling Fluid Systems

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ME584

Agenda

- Introduction
- Properties of Fluid and Reynolds Number Effects
- Passive Components
- Case Study: Spring-Loaded Diaphragm Actuator

- Active Learning: Pair-share Exercises, Case Study

Introduction

- Fluid Systems
  - Operate through effects of either liquids or gases
  - Have wide range of applications, e.g., vehicle suspension systems, hydraulic servomotors, and chemical processing systems
- Hydraulics (fluid is incompressible) and pneumatic (fluid is compressible) systems
  - Common modeling principle is conservation of mass
  - Key advantages relative to electro-mechanical systems
    - Power density of pump/actuators (1 order of mag. higher 200 psi electromagnetic actuator vs. 3000-8000 psi hydraulic actuators)
    - Circulating fluid removes heat generated by actuator (instead of free or forced convection)
  - Have more nonlinearities -> challenging for modeling and simulation

Properties of Fluid

Fluid Density

- Incompressible: density of fluid (e.g., liquid) remains constant despite changes in fluid pressure (an approximation, but simpler modeling)
- Compressible: density of fluid (e.g., gas) changes with pressure
- Liquids have higher density, absolute viscosity, bulk modulus, and exhibits surface tension effects
- Density of a fluid: mass m per unit volum V under pressure P_0 and temperature T_0

$$\rho = \frac{m}{V}\bigg|_{P_0,\, T_0}$$

Equation of State: Liquids

- Equation of state: relationship between *ρ,P, and T :*

$$\rho = \rho_0 + \left.\frac{\partial \rho}{\partial P}\right|_{P_0,\,T_0}(P - P_0) + \left.\frac{\partial \rho}{\partial T}\right|_{P_0,\,T_0}(T - T_0)$$

Reference conditions:
$\rho_0, P_0, T_0,$

- Substitute with Bulk modulus β and Coefficient of thermal expansion α

$$\rho = \rho_0\left[1 + \frac{1}{\beta}(P - P_0) - \alpha(T - T_0)\right]$$

where

$$\beta = \rho_0 \left.\frac{\partial P}{\partial \rho}\right|_{P_0,\,T_0} = \left.\frac{\partial P}{\partial \rho/\rho_0}\right|_{P_0,\,T_0}$$

$$\alpha = -\left.\frac{1}{\rho_0}\frac{\partial \rho}{\partial T}\right|_{P_0,\,T_0}$$

Equation of State: Liquids

• Isothermal bulk modulus: pressure changes occur at a slow enough rate during heat transfer to maintain constant temperature	$$\beta = -\frac{\partial P}{\partial V/V_0}$$
• Adiabatic bulk modulus: pressure change is significant, preventing heat transfer. Specific heats ratio C_p/C_v ~ 1	$$\beta_a = \frac{C_p}{C_v}\beta$$ (C_p and C_v specific heat at constant pressure and temperature)
• Thermal expansion coefficient: incremental change in volume with temperature changes ~ $0.5 \times 10^{-3}/^{0}F$ for most liquid	$$\alpha = \left.\frac{\partial V/V_0}{\partial T}\right\|_{P_0,\, T_0}$$

Equation of State: Gases

- Ideal gas: $$\rho = \frac{P}{RT}$$

- Gas undergoing polytropic process: $$\frac{P}{\rho^n} = C = \text{ constant}$$

where

$n = 1.0$ for an isothermal process

$n = k$ for an adiabatic process (k = ratio of specific heats)

$n = 0.0$ for an isobaric process

$n = \infty$ for an isovolumetric process

- Bulk modulus:

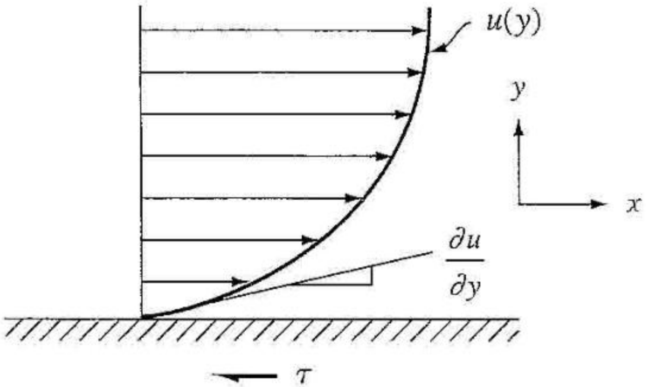
$$\beta = \rho_0 \frac{\partial P}{\partial \rho}\bigg|_{P_0,\,T_0} = \rho_0[nC\rho^{n-1}]\bigg|_{P_0,\,T_0} = \rho_0 n \frac{C\rho^n}{\rho}\bigg|_{P_0,\,T_0} = nP_0$$

(β_{liquid} ~ 5 to 15 Kbar >> β_{gas} ~ 1 to 10 Bar)

Viscosity

- Absolute viscosity μ

$$\mu = \frac{\text{shear stress}}{\text{shear rate}} = \frac{\tau}{\partial u / \partial y}$$



- Kinematic viscosity: $\nu = \mu/\rho$
- Liquids: λ_L constant depends on liquid

$$\mu = \mu_0 e^{-\lambda_L (T - T_0)}$$

- Gases: λ_G constant depends on gas

$$\mu = \mu_0 + \lambda_G (T - T_0)$$

Speed of Sound, Specific Heat Ratio, and Reynolds Number

- Speed of sound or propagation

$$c_0 = \sqrt{\frac{\beta}{\rho}}$$ ~ 1370 m/s in oil at 25^0 C, $$c_0 = \sqrt{kRT}$$ ~ 347 m/s in air at 25^0C

- Specific ratio $k = \frac{C_p}{C_v}$ ~ 1.4 for gases and 1.04 for liquids

- Reynolds Number: inertial forces / viscous forces

$$N_r = \frac{(\rho d^2 v^2)}{(\mu d v)} = \frac{vd}{\nu}$$

Laminar: $N_r < 1400$, transition: $1400 < N_r < 3000$, turbulent: $N_r > 3000$

Passive Components:
Capacitance, Inductance, Resistance

BASIC ELEMENTS IN ENGINEERING DISCIPLINES.

	Dissipative (Resistive)	Effort Storage (Capacitive)	Flow Storage (Inductive)
Mechanical Translation Effort = Force Flow = Velocity	$F = b(v_1 - v_2)$	$F = \frac{k}{D}(v_1 - v_2)$	$F = mDv_1$
(Alternative Form) Effort = Force Flow = Position	$F = bD(x_1 - x_2)$	$F = k(x_1 - x_2)$	$F = mD^2x_1$
Mechanical Rotation Effort = Torque Flow = Speed	$T = b(\omega_1 - \omega_2)$	$T = \frac{k}{D}(\omega_1 - \omega_2)$	$T = JD\omega_1$
(Alternative Form) Effort = Torque Flow = Angle	$T = bD(\theta_1 - \theta_2)$	$T = k(\theta_1 - \theta_2)$	$T = JD^2\theta_1$
Electrical Effort = Voltage Flow = Current	$e_1 - e_2 = Ri$	$e_1 - e_2 = \frac{1}{CD}i$	$e_1 - e_2 = LDi$
Fluid Effort = Pressure Flow = Volume Flow Rate	$P_1 - P_2 = RQ$	$P_1 = \frac{1}{CD}Q$	$P_1 - P_2 = LDQ$
Thermal Effort = Temperature Flow = Heat Flow	$T_1 - T_2 = RQ_h$	$T_1 = \frac{1}{CD}Q_h$	Does Not Exist

Fluid Capacitance

Conservation of mass for a control volume (cv)

$$\dot{m}_{net} = \frac{d}{dt}(M_{cv}) = \frac{d}{dt}(\rho_{cv} V_{cv})$$

$$\dot{m}_{net} = \rho Q_{net} = \rho_{cv}\dot{V}_{cv} + V_{cv}\dot{\rho}_{cv}$$

If all densities (inlet, outlet, cv) are constant and equal to ρ,

$$Q_{net} = \dot{V} + \frac{V}{\rho}\dot{\rho}$$

but $\rho = \rho(P,T)$ and $\beta = \rho_0 \partial P / \partial\rho \,|_{P_0,T_0}$,

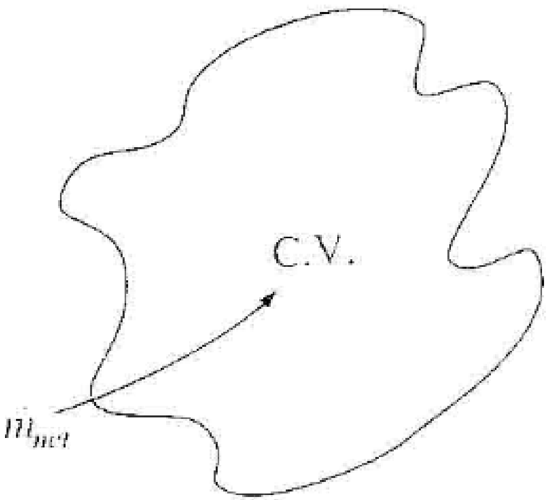
$$\dot{\rho} = \frac{\partial\rho}{\partial P}\frac{\partial P}{\partial t} = \frac{\rho_0}{\beta}\dot{P}$$

Substitute into the flow equation,

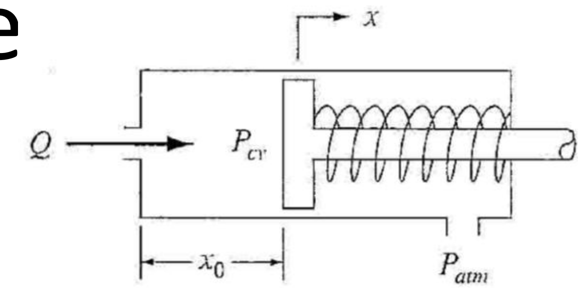
$$Q = \dot{V} + \frac{V}{\beta}\dot{P}_{cv} \quad or \quad \dot{P}_{cv} = \frac{\beta}{V}(Q - \dot{V}) \text{ (continuity equation)}$$

when $\dot{V} = 0$, $Q = \frac{V}{\beta}\dot{P}_{cv}$

$Q = C_f \dot{P}_{cv}$, where $\mathrm{C_f} = \frac{V}{\beta}$ = capacitance for large volume of compressible fluid.



Example 1: Spring-Loaded Piston Capacitance



- Find total capacitance for this system

Assume inertial and frictional effects are negligible ,

(Is this a good assumption?)

$$Force\ balance:\ A\delta P - kx = 0,\ \ where\ \delta P = P_{cv} - P_{atm}$$

$$\dot{x} = A\delta\dot{P}/k$$

$$Cylinder\ volume\ V = A(x + x_0)\ or\ \dot{V} = A\dot{x} = A^2\delta\dot{P}/k$$

$$Q = \dot{V} + \frac{V}{\beta}\dot{P}_{cv} = \left(\frac{V}{\beta} + \frac{A^2}{k}\right)\delta\dot{P} = C_f\,\delta\dot{P}$$

$$Q = \left(\frac{V}{\beta} + \frac{A^2}{k}\right)\delta\dot{P}$$

Mechanical capacitance

Compressibility capacitance

(β_{liquid} ~ 10 Kbar >> β_{gas} ~ 10 Bar)

- Accumulators: liquid capacitors
  - Spring-loaded pistons, bellows, gas-filled bladders
  - Use mechanical capacitors when β is large for incompressible fluids
  - Use large V to get large compressibility capacitance

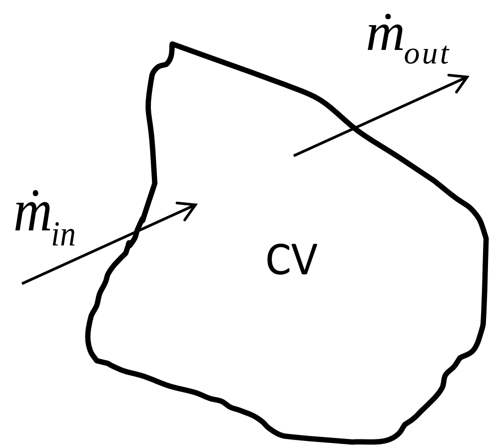
Capacitance for Gases

For a gas with significant temperature or pressure difference between inlet gas and cv, energy equation is used (neglect kinetic and potential energy)

$$q_{net} + C_p(\dot{m}_{in}T_{in} - \dot{m}_{out}T_{out}) - \dot{W} = \frac{d}{dt}(U_{cv})$$

where q_{net} = heat transfer between surroundings and cv;

U = internal energy; W = work



Algebraic manipulations result in

$$\dot{P}_{cv} = \frac{nP_{cv}}{V_{cv}}\left[\sum \frac{\dot{m}_{in}}{\rho_{cv}}\left\{\frac{k(1-r)\frac{T_{in}}{T_{cv}} + r}{k(1-r)+r}\right\} - \frac{\dot{m}_{out}}{\rho_{cv}} - \dot{V}_{cv}\right]$$

where $n = k(1\text{-}r) + r$; $r = \dfrac{q_{net}}{q_{\max}}$; $\rho_{cv} = \dfrac{P_{cv}}{RT_{cv}}$

$q_{\max}$ = maximum heat transfer required to make inlet flow temperature equal cv temperature;

Example 2: Capacitance of Thin-Walled Tube

- A circular tube of length l is used to hold fluid pressure. If the tube has an internal diameter d_i, a wall thickness t, and a Young's modulus E,
  - Derive the capacitance of the tube, using an incompressible fluid,
  - Derive the total capacitance C_T, which includes the volume capacitance of the fluid, C_F (with a fluid of bulk modulus β), and the mechanical capacitance, C_M.

Example 2: Capacitance of Thin-Walled Tube

The Hoop stress in a thin-walled tube of finite length is given by the following.

$$\sigma = \frac{F}{A} = \frac{d_i \, \ell \, \delta P_i}{2 t \ell}$$

From Hooke's equation, the strain (change in circumference divided by original circumference) is related to the stress.

$$\sigma = E \varepsilon = E \frac{\pi \delta d_i}{\pi d_i}$$

Thus, by equating these last two equations for stress, we can see that the pressure is proportional to the change in diameter as follows.

$$\frac{\delta d_i}{d_i} = \frac{1}{E} \frac{d_i \, \delta P_i}{2t} \qquad \text{or} \qquad \delta d_i = \frac{d_i^2 \, \delta P_i}{2 E t}$$

The volume inside the tube can be expressed as follows. $V = \frac{\pi}{4}(d_i + \delta d_i)^2 \ell$

where the initial volume is $V_0 = \frac{\pi}{4} d_i^2 \ell$

The derivative of the volume is $\dot{V} = \frac{2 \pi \ell}{4}(d_i + \delta d_i) \, \delta \dot{d}_i$

Example 2: Capacitance of Thin-Walled Tube

From a previous equation for δd_i $$\delta \dot{d}_i = \frac{d_i^2}{2Et}\delta \dot{P}_i$$

Thus, the volume derivative is $$\dot{V} = (d_i + \delta d_i)\frac{\pi d_i^2 \ell}{4Et}\delta \dot{P}_i = (d_i + \delta d_i)\frac{V_0}{Et}\delta \dot{P}_i$$

Which, if the change in diameter is small, reduces to $$\dot{V}_i = \frac{V_0 d_i}{Et}\delta \dot{P}_i$$

The continuity equation is $$Q = \dot{V} + \frac{V_0}{\beta}\delta \dot{P}_i$$

Substituting the volume derivative expression $$Q = \frac{V_0 d_i}{Et}\delta \dot{P}_i + \frac{V_0}{\beta}\delta \dot{P}_i$$

After rearrangement, yields the final result $$Q = \frac{V_0}{\beta}\left[1 + \frac{\beta d_i}{Et}\right]\delta \dot{P}_i$$

The total capacitance from this result is $$C_t = \frac{V_0}{\beta}\left[1 + \frac{\beta d_i}{Et}\right]_i$$

If the tube had been extremely stiff, then the capacitance of the fluid alone is $C_f = \dfrac{V_0}{\beta}$

The mechanical capacitance is the term $$C_m = \frac{V_0 d_i}{Et}$$

Example 3: Pair-Share: Capacitance of a Balloon

The radius expansion, $R-R_0$, of a balloon filled with a gas is directly proportional to the internal pressure of the gas. Let us write this proportionality as $\delta P=K(R-R_0)$. Derive an expression for the total capacitance of the balloon that considers the change in volume of the balloon and the effect of compressibility of the gas. (Volume = $4\pi R^3/3$)

Example 3: Pair-Share: Capacitance of a Balloon

If we assume that air is introduced at approximately the same temperature of the air inside the balloon, then we can use the simplified equation for the continuity equation, Eq. 5.27.

$$Q = \dot{V} + \frac{V}{\beta}\dot{P}_{cv}$$

The volume of a spherical balloon is $V = \frac{4}{3}\pi R^3$ thus, $\dot{V} = 4\pi R^2 \dot{R}$

The internal pressure of the balloon is $\delta P_i = k(R - R_0)$ thus, $\delta \dot{P}_i = k\dot{R}$

from the continuity equation $Q = 4\pi R^2 \dot{R} + \frac{V}{\beta}k\dot{R} = \left[\frac{3}{R}V + \frac{V}{\beta}k\right]\dot{R}$

rearranging $Q = \frac{V}{\beta}\left[\frac{3\beta}{R} + k\right]\dot{R} = \frac{V}{\beta}\left[1 + \frac{3\beta}{k R}\right]\delta\dot{P}_i$

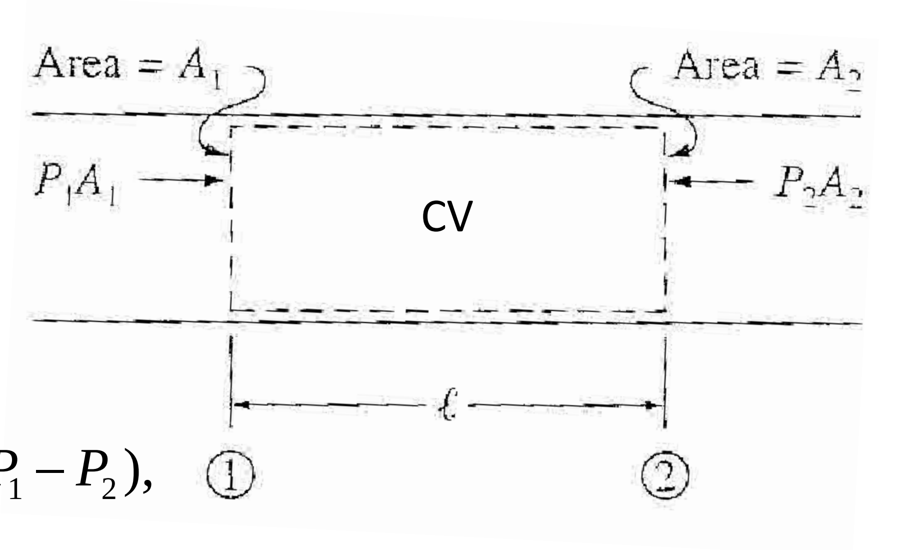
Therefore, the total capacitance is $C_{total} = \frac{V}{\beta}\left[1 + \frac{3\beta}{k R}\right]$

Fluid Inductance

Consider cv with constant mass ($\dot{m}=0$)

$$\sum \mathrm{F}_{\text{ext}} = \frac{d}{dt}(mv)_{cv} = m\dot{v} + \dot{m}v = m\dot{v}$$

$$\sum \mathrm{F}_{\text{ext}} = P_1 A_1 - P_2 A_2 = m\dot{v}$$



Let $A = A_1 = A_2;\ \ m = \rho A l;\ Q = Av; \delta P = (P_1 - P_2),$

$$(P_1 - P_2)A = \rho A l \, \frac{d}{dt}(v)_{cv}$$

$$or\ \ \delta P = \frac{\rho l}{A}\dot{Q}$$

Fluid inductance is $\dfrac{\rho l}{A}$, valid for liquids and

gases (for gases, density must be evaluated at upstream conditions)

Fluid Resistance

- Laminar flow: viscous-dominated flow
  - Low enough flow rates or pressure drop in long capillary tubes -> viscous flow
  - Viscous terms dominate -> Reynolds is low ($N_r < 1000$)
- Orifice-type or head loss resistance: inertia-dominated flow
  - Orifice with short length in direction of flow
  - Head loss with turbulent flow
- Compressible flow resistance
  - Similar to orifice type, but includes density variation of gas
  - Flow equations have high degree of nonlinearity

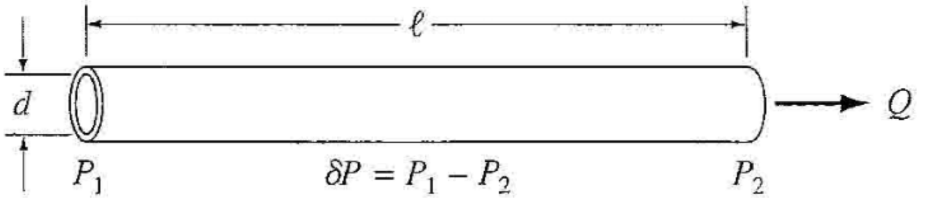
Laminar Flow Resistance

$$\delta P = RQ$$

with hydraulic diameter d_h; absolute viscosity μ; length l; area A

$$R = \frac{32\mu l}{A d_h^2} \quad \text{(General resistance)}$$

$$d_h = \frac{4\,area}{perimeter}$$

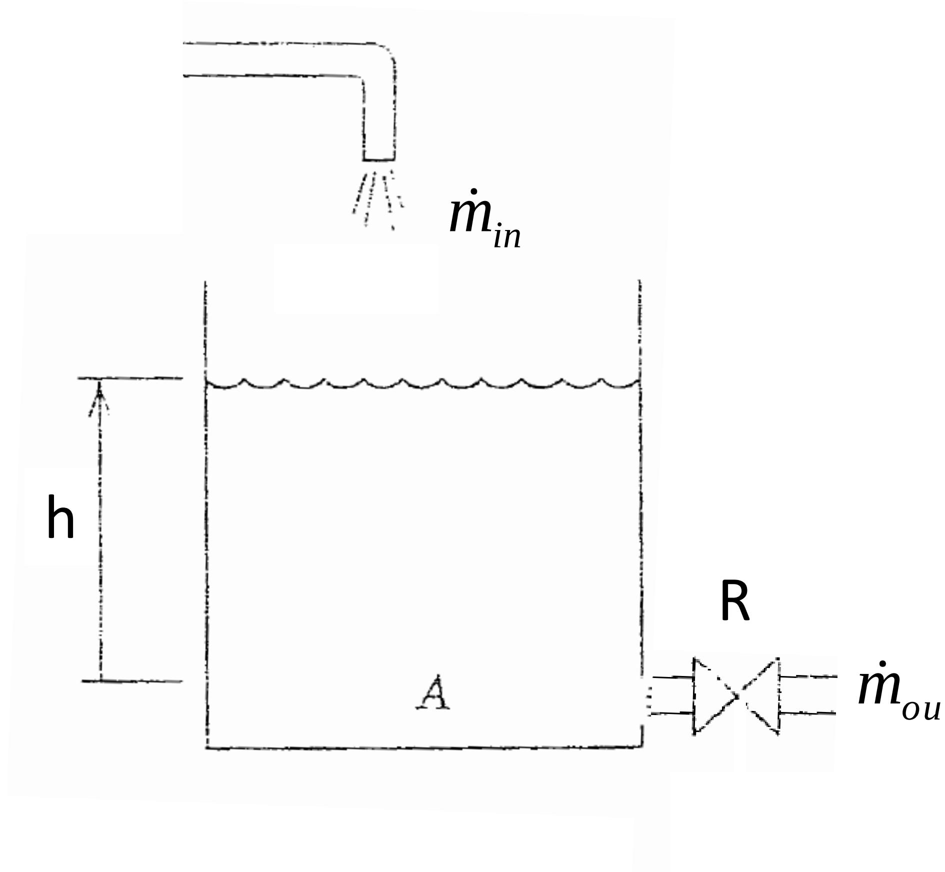


$$R = \frac{128\mu l}{\pi d^4} \quad \text{(Circular section)}$$

$$R = \frac{8\mu l}{\dfrac{wh^3}{(1 + h/w)^2}} \quad \text{(Rectangular section)}$$

Example 4: A Liquid-Level System

The tank shown has a mass inflow rate of $\dot{m}_{in}$. The liquid height above the orifice is h. Compute the time constant of the system, assuming that the flow is laminar. The tank contains fuel oil at 70^0F with a mass density ρ of 1.82 slug/ft^3 and a viscosity μ = 0.02 lb-sec/ft^2. The outlet pipe diameter D is 1 in., and its length L is 2 ft. The tank is 2 ft in diameter.



Example 4: A Liquid-Level System

From conservation of mass and the laminar flow resistance relation,

$$\dot{m}_{\text{tank}} = \dot{m}_{in} - \dot{m}_{out}$$

$$\rho A \frac{dh}{dt} = \dot{m}_{in} - \rho Q_{out} \ \ or \ \ \rho A \frac{dh}{dt} = \dot{m}_{in} - \rho \frac{\delta P}{R}$$

$$but \ \ \delta P = \rho g h \ \ and \ \ R = \frac{128 \mu l}{\pi D^4},$$

$$\rho A \frac{dh}{dt} = \dot{m}_{in} - \frac{\rho^2 g h}{R} \ \ or \ \ \frac{dh}{dt} + \frac{\rho g}{RA} h = \frac{1}{\rho A} \dot{m}_{in}$$

$$but \ \ \rho \frac{1}{R} g \frac{1}{A} = (1.82) \frac{\pi (1/12)^4}{128(0.02)(2)} (32.2) \frac{1}{\pi} = 1812 \text{ sec}$$

$$\frac{dh}{dt} + 1812 h = \frac{1}{\rho A} \dot{m}_{in}$$

$$\tau = 1812 \text{ sec} \sim 30 \text{ minutes}$$

Orifice Flow Resistance

Apply Bernoulli's equation for streamlines between 1 and 2, 2 and 3,

$$\frac{P_1}{\rho}+\frac{v_1^2}{2}=\frac{P_2}{\rho}+\frac{v_2^2}{2}=\frac{P_3}{\rho}+\frac{v_3^2}{2}$$

Equating volume flow rate at all cross sections,

$$Q=A_1v_1=A_2v_2=A_3v_3=A_4v_4$$

Assume sudden expansion between 2 and 4, then there is a velocity head loss and pressures are same at 3 and 4, hence $P_3=P_4$ (easy to measure)

Assume $A_1>A_2$ and $A_1>A_3$, then $v_1<v_3$ and $v_4 < v_3$, so v_1 and v_4 are negligible

Combine above equations with assumptions,

$$P_1-P_4=\frac{\rho}{2}v_3^2$$

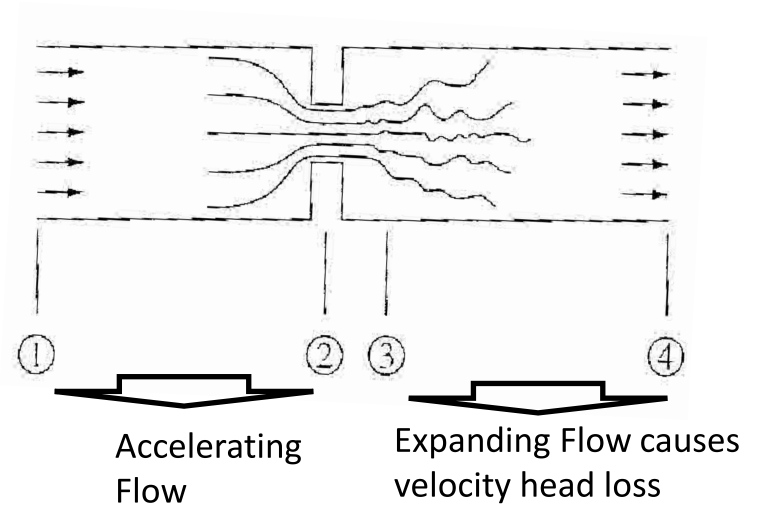
Use discharge coefficient C_d $\left(\begin{array}{l}\text{combined effects of losses betwen 1 and 3,}\\ \text{and representing } A_3 \text{ in term of } A_2\end{array}\right)$

$$\delta P=P_1-P_4=\frac{\rho}{2C_d^2A_2^2}Q^2 \qquad (\textit{orifice flow relation})$$

$$or\ Q=C_dA_2\sqrt{\frac{2\delta P}{\rho}}=\sqrt{\frac{|\delta P|}{K}}sign(\delta P)$$

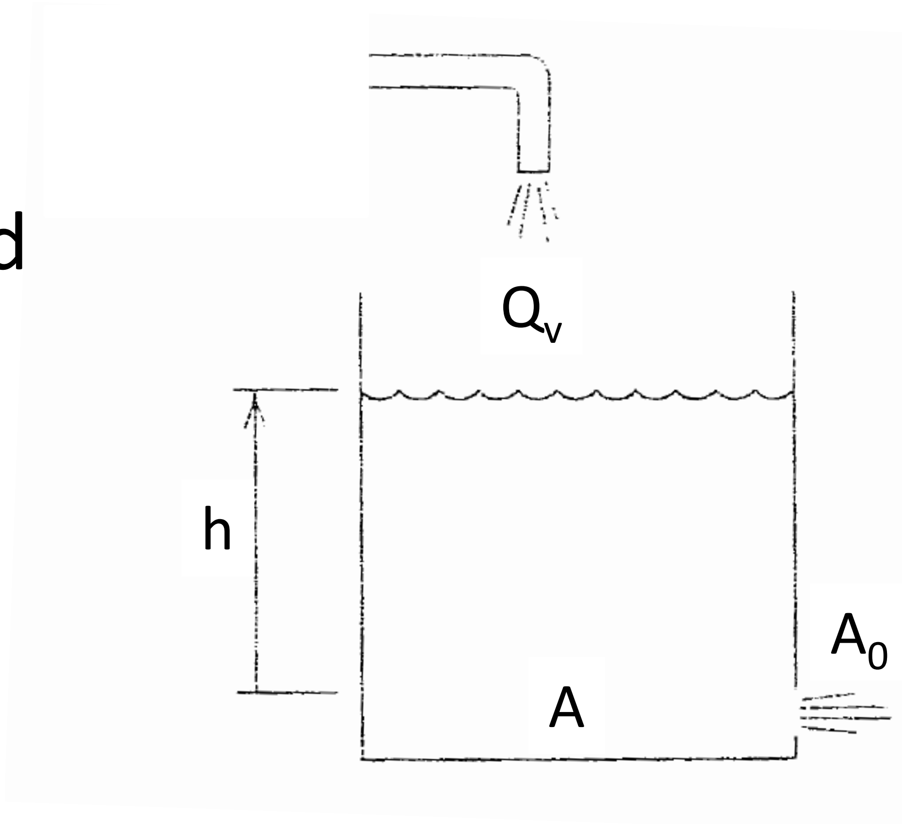
$$where\ \ K=\frac{\rho}{2C_d^2A_2^2};\ sign(x)=1\ if\ \ x>0;\ sign(x)=-1\ if\ \ x<0$$

$$C_d=C_d\,(geometry, roughness, orifice\ configuration, N_r)\sim .5\ to\ 1$$



Example 5: Tank with an Orifice

- The tank shown has an orifice in its side wall. The orifice area is A_0 and the bottom area of the tank is A. The liquid height above the orifice h. The volume inflow rate is Q_v. Develop a model of the height h with Q_v as the input.



Example 5: Tank with an Orifice

From conservation of mass and the orifice flow relation,

$$\dot{m}_{\text{tank}} = \dot{m}_{in} - \dot{m}_{out}$$

$$\rho A \frac{dh}{dt} = \rho Q_v - \rho C_d A_0 \sqrt{\frac{2\delta P}{\rho}}$$

$$but\ \delta P = \rho g h,$$

$$\rho A \frac{dh}{dt} = \rho Q_v - \rho C_d A_0 \sqrt{2gh}$$

This is a nonlinear equation, how can it be analyzed?