Modeling Electrical Systems

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ME584

Agenda

- Basic Electrical Elements
- Passive Circuit Analysis
- Active Circuit Analysis
- Case Study: A Speaker Model

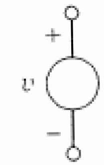
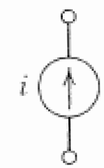
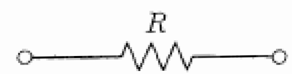
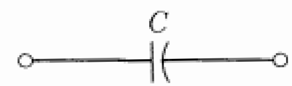
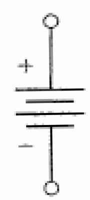
- Active Learning: Pair-share Exercises, Case Study

Basis Electrical Elements

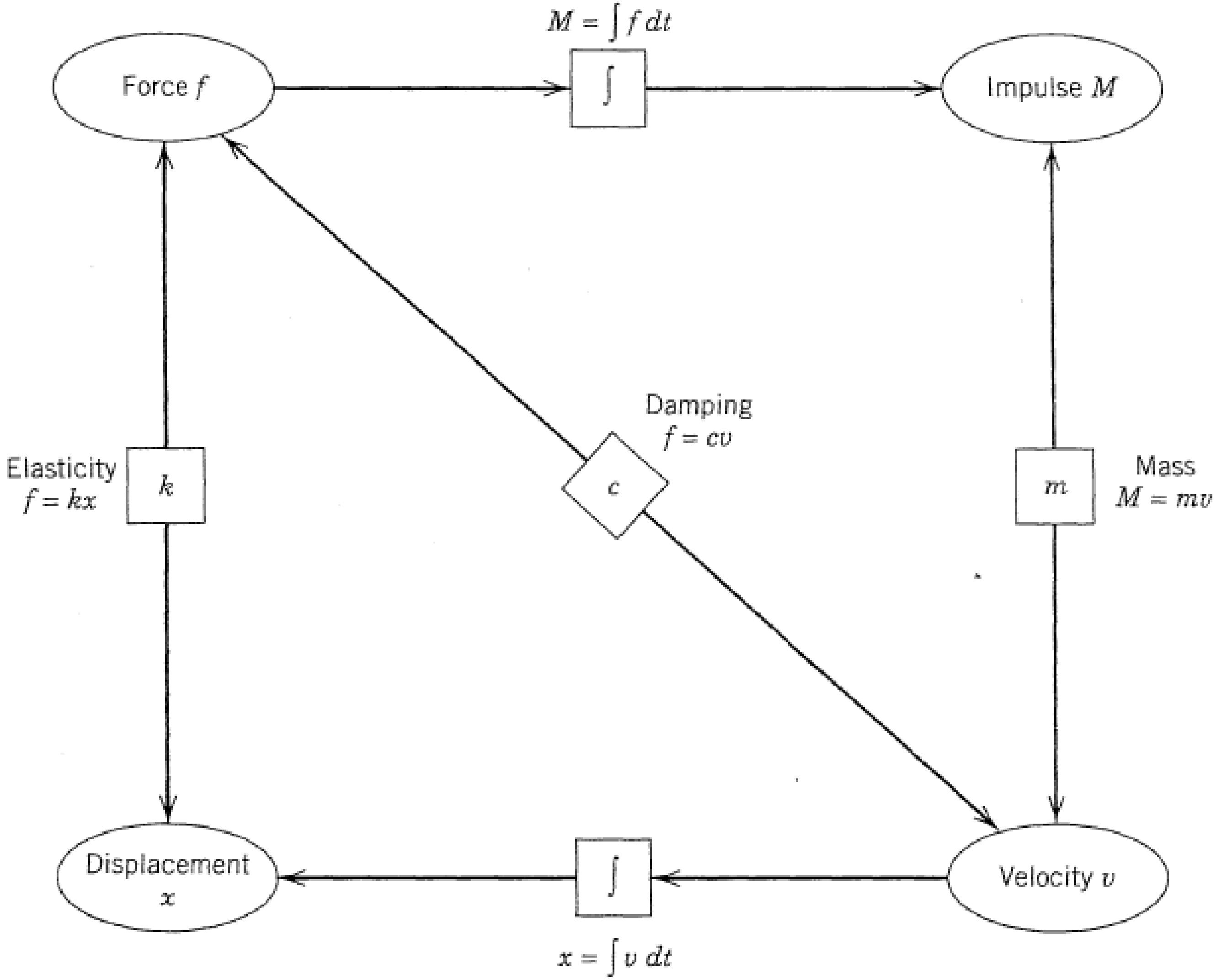
Electrical System

- Composed of resistors, capacitors, inductors, transistors, amplifiers, power supplies
  - Passive circuits: respond to applied voltage or current and do not have any amplifiers
  - Active circuits: made of transistors and/or amplifiers, require active power source to work
- Basic quantities
  - Charge q [coulomb] = 6.24×10^{18} electrons
  - Current i [ampere] = dq/dt
  - Voltage e [Volt] = dw/dq
  - Energy or Work w [joule]
  - Power p [watt] = e x i = dw/dt

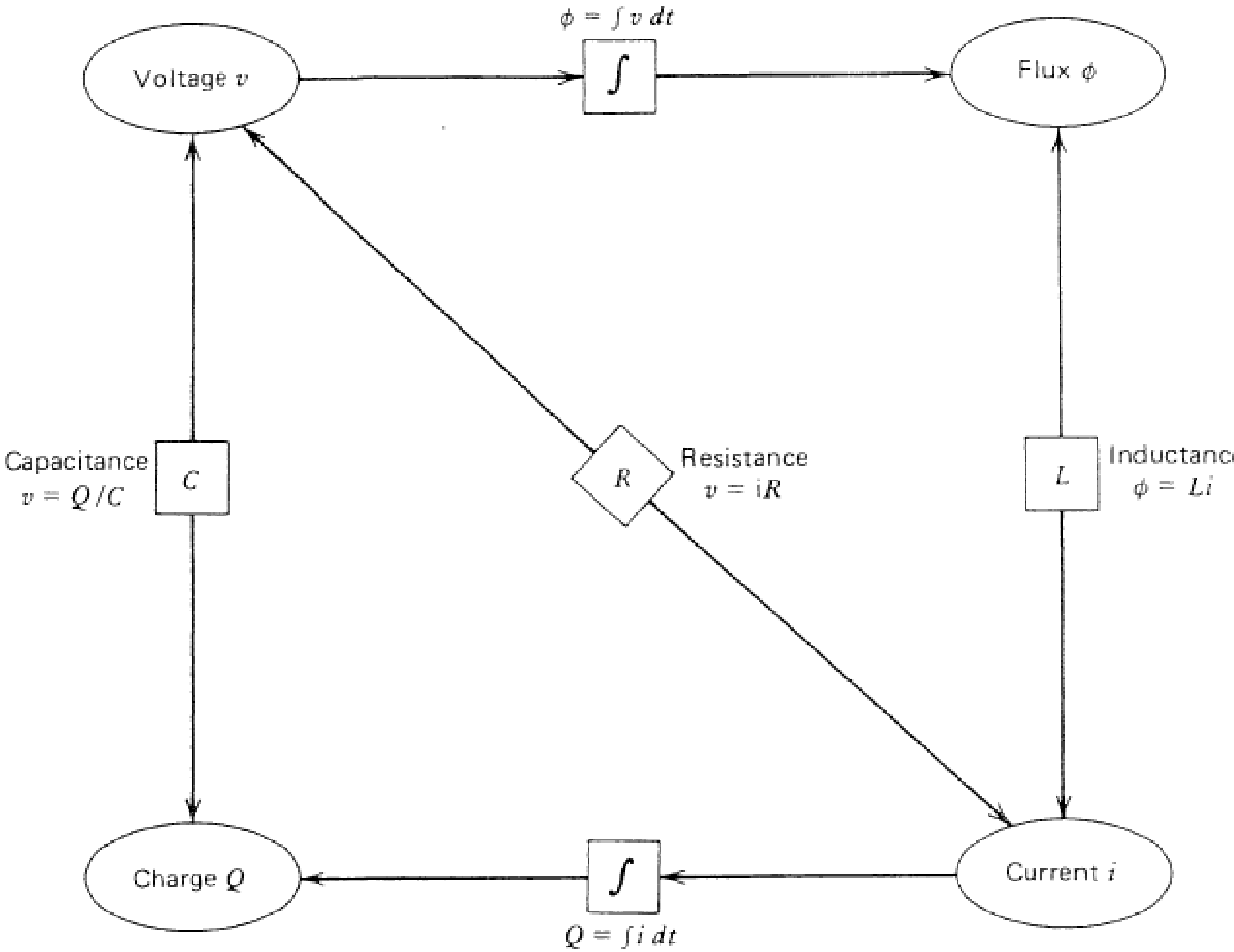
Units and Representations for Common Electrical Quantities

Qunatity and Symbol	Units	Representation
Voltage v	volt (V)	Voltage source 
Charge Q	coulomb = newton-meter/volt	
Current i	ampere (A) = coulomb/second	Current source 
Flux ϕ	volt-second	
Resistance R	ohm (Ω) = volt/ampere	
Capacitance C	farad (F) = coulomb/volt	
Inductance L	henry (H) = volt-second/ampere	
Battery (voltage source plus internal resistance)	–	
Ground (zero voltage reference)	–	

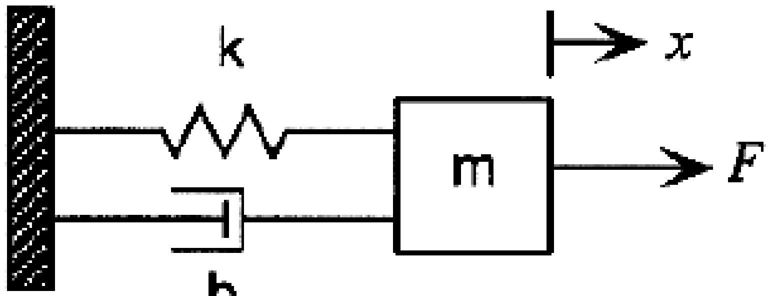
General Model Structure for Mechanical System



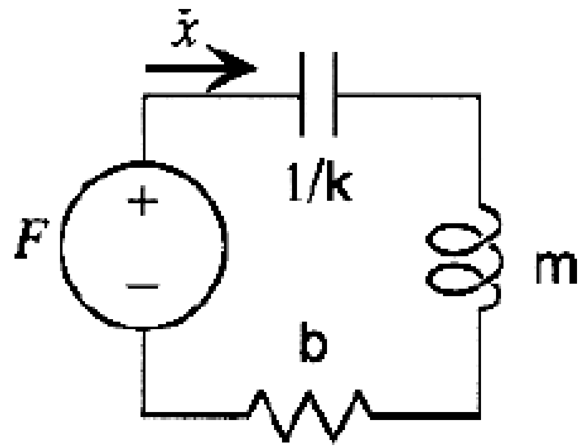
General Model Structure for Electrical System



Equivalent Circuit for Spring-Mass-Dashpot Systems



Spring-mass-dashpot system



Equivalent circuit

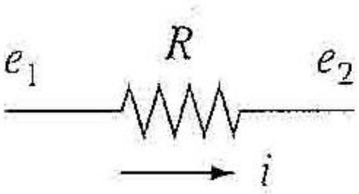
(All 3 elements share the same displacement) ⟶ Series Connection

Connection rule for the equivalent circuit for e → V convention:

- **Elements share a common Flow or Displacement → Connected in Series**
- **Elements share a common Effort → Connected in Parallel**

Resistance

- Resistance behavior is between insulator and conductors, allowing a predictable restriction of electron flow



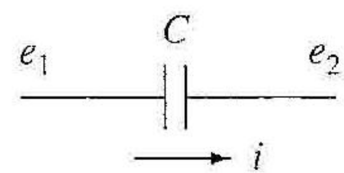
$$\delta e = Ri \qquad \text{where } \delta e = e_1 - e_2$$

- Power dissipated = $\delta e\, i$
- Resistance $R = \dfrac{\rho l}{A}$
  - A: cross section are of wire
  - l: length of wire
  - ρ: resistivity of material

Capacitance

- Capacitor stores electrons on 2 parallel plates separated by an insulating dielectric material in an electric field

$$\delta e = \frac{1}{C}\int_{-\infty}^{t} i\,dt \quad \text{or} \quad i = C\frac{d\,\delta e}{dt}$$



- Energy stored in capacitor

$$w = \int_{-\infty}^{t} \delta e\, i\,dt = \int_{-\infty}^{t} \delta e\left(C\frac{d\,\delta e}{dt}\right)dt = \int_{0}^{\delta e} C\,\delta e\,d\,\delta e = \frac{C\,\delta e^2}{2}$$

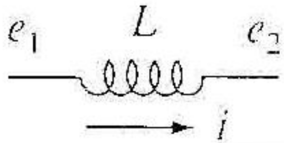
- Capacitance $C = \frac{\epsilon A}{d}$
  - A: area of plates
  - D: spacing between plates
  - ϵ:permittivity of the dielectric

Inductance

- Inductance relates voltage induced to time rate of change of magnetic field
- Faraday's law:

$$e_i = \frac{d\phi}{dt}$$ where ϕ is the magnetic flux, ϕ = Li

$$\delta e = L\frac{di}{dt} \qquad \text{where } \delta e = e_1 - e_2$$



- Energy stored $w = \int_{-\infty}^{t} \delta e\, i\, dt = \int_{-\infty}^{t}\left(L\frac{di}{dt}\right) i\, dt = \int_{0}^{i} Li\, di = \frac{Li^2}{2}$
- Inductance:

$$L = \frac{\mu_m n^2}{\ell} A$$ where A= wire cross section area, l = wire length, n = number of turns, μ_m =permeability of magnetic circuit

Impedance

- Impedance Z: instantaneous ratio of voltage difference to current

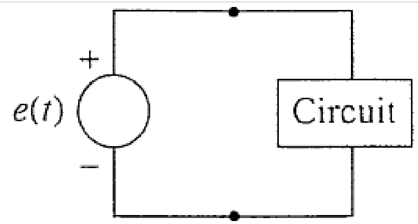
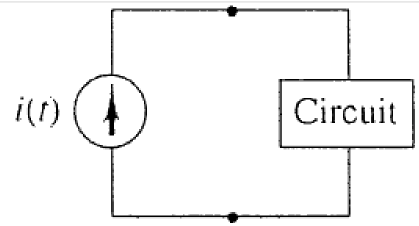
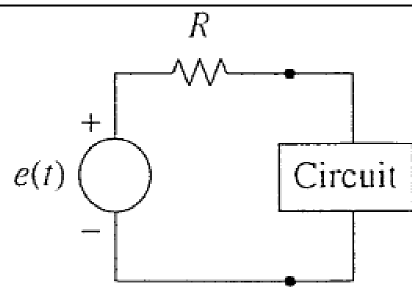
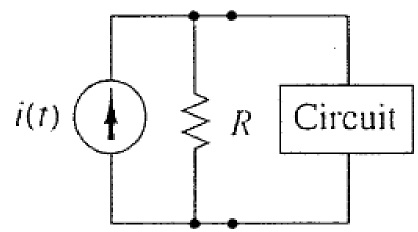
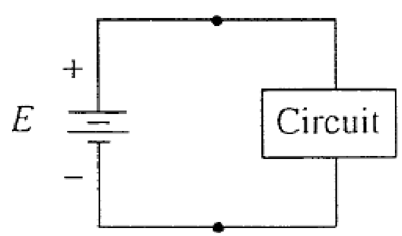
$$Z = \frac{\delta e}{i}$$

- Impedance of common circuit elements
  - Resistive: $Z_r = R$ (not dynamic)
  - Capacitive: $Z_c = 1/CD$
  - Inductive: $Z_L = LD$

  Where D= differential operator d/dt,

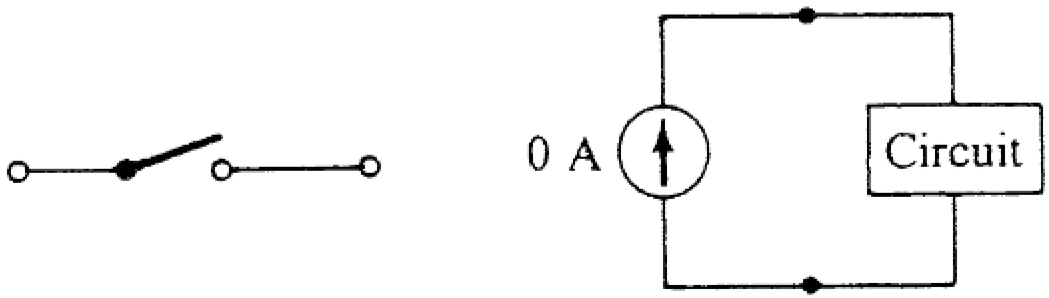
  1/D = integrator operator

Ideal and Non-Ideal Sources

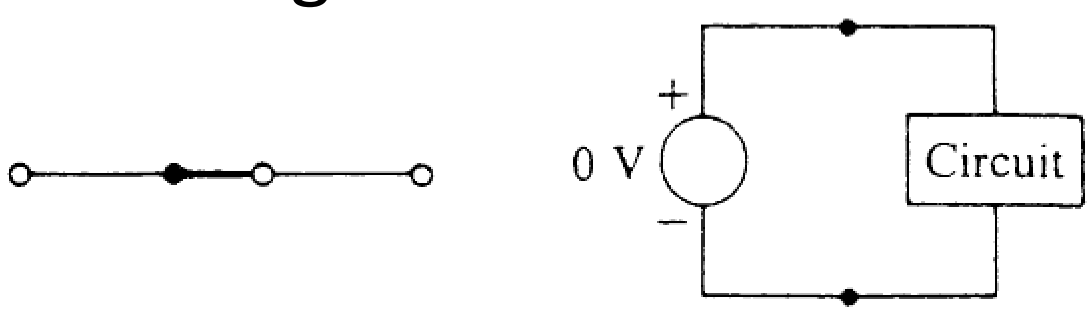
	Voltage Source	Current Source
Ideal	 (a)	 (b)
Non-Ideal	 (c)	 (d)
Battery	 (e)	

Open and Short Circuits

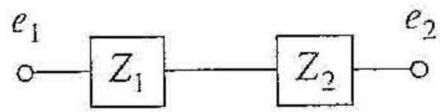
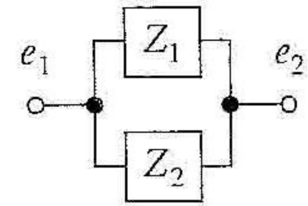
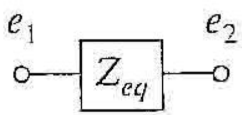
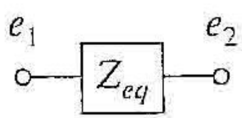
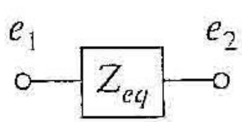
- An open circuit is any element through which current cannot flow



- A short circuit is any element across which there is no voltage

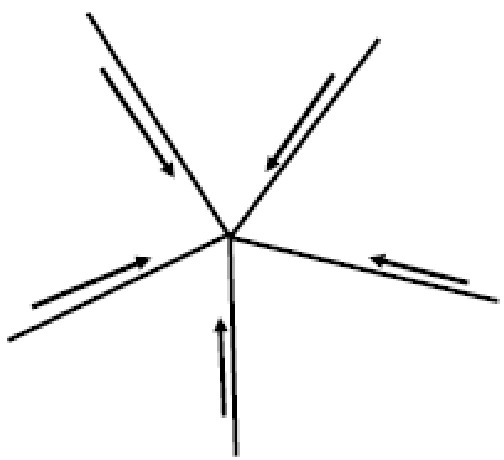


Series and Parallel Impedance Combinations

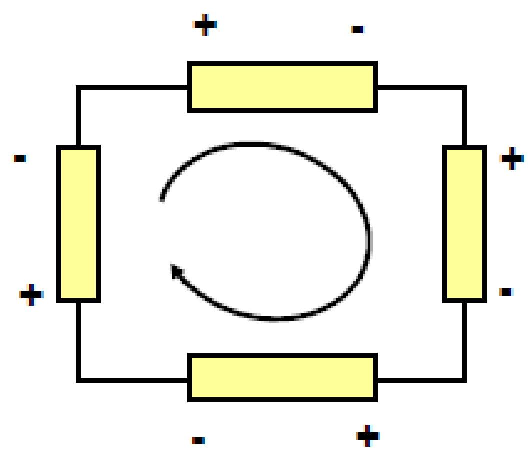
		 Series	 Parallel
R	 Equivalent	$R_{eq} = R_1 + R_2$	$R_{eq} = \dfrac{R_1 R_2}{R_1 + R_2}$
C	 Equivalent	$C_{eq} = \dfrac{C_1 C_2}{C_1 + C_2}$	$C_{eq} = C_1 + C_2$
L	 Equivalent	$L_{eq} = L_1 + L_2$	$L_{eq} = \dfrac{1}{L_1} + \dfrac{1}{L_2}$

Laws for Passive Circuit Analysis

- Kirchhoff's Current Law (KCL)
  - The sum of all currents (flow) entering a node is zero
- Kirchhoff's Voltage Law (KVL)
  - The oriented sum of all voltages (efforts) around any closed loop is zero



KCL



KVL

Techniques for Passive Circuit Analysis for Classical Deriving Differential Equations

1. Draw circuit schematic and label components (e.g., R_1, R_2, C_1, L_1…)
2. Assign voltage at each node (e.g., e_1, e_2)
3. Assign current in each component (e.g., i_1, i_2, ..) and show positive current direction with arrows
4. Write equation for current for each component (e.g., $i_{R1} = (e_1 - e_2)/R_1$ or $i_{C1} = CDe_1$)
5. Write node equations for each significant node (not connected to voltage or current source)
6. Substitute component equations into node equations and reduce results to a single differential equation with output and input variables

Example 1: Voltage Divider

Evaluate e_1

$$i_1 = \frac{e_0 - e_1}{R_1}$$

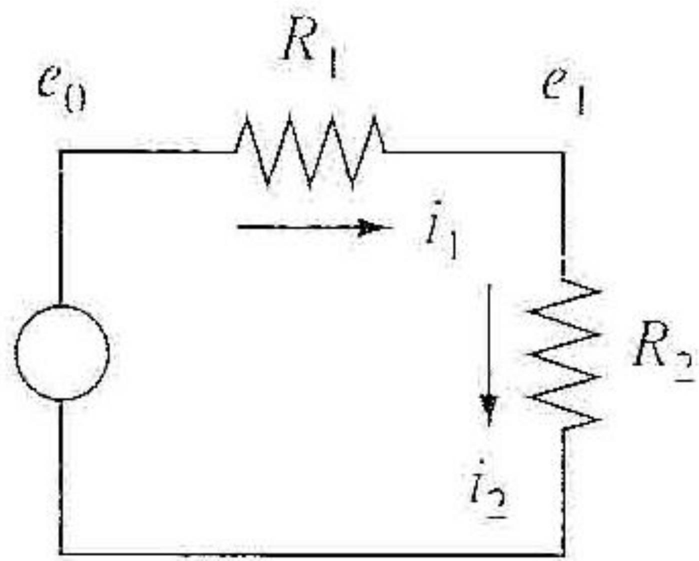
$$i_2 = \frac{e_1 - 0}{R_2}$$

$$i_1 - i_2 = 0$$

$$\frac{e_0 - e_1}{R_1} - \frac{e_1}{R_2} = 0$$

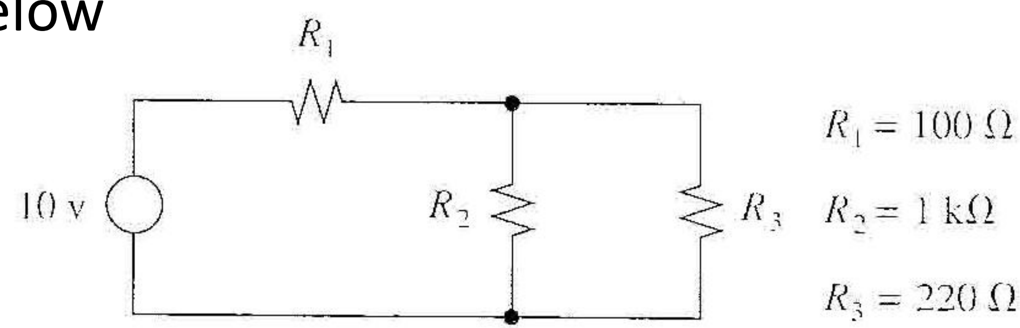
$$e_1 = \frac{R_2}{R_1 + R_2} e_0$$

$$e_1 = \frac{1}{1 + \frac{R_1}{R_2}} e_0$$



Example 2: Resistor Circuit

- Calculate the amount of power dissipated in resistor R_3 in the circuit shown below



- Solution:

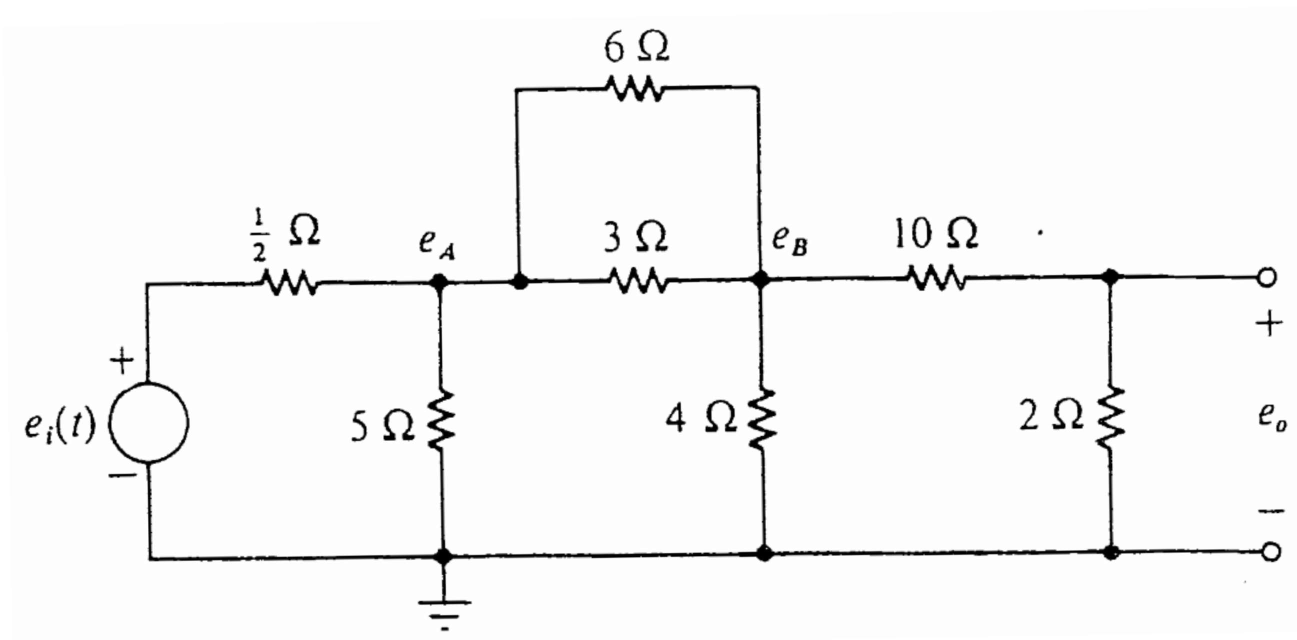
resistors in parallel $$R_{eq} = \frac{R_2 R_3}{R_2 + R_3} = \frac{1000 \times 220}{1000 + 220} = 180.3\ \Omega$$

voltage divider equation $$e_1 = \frac{R_{eq}}{R_1 + R_{eq}} e_0 = \frac{180.3}{100 + 180.3} 10 \text{ volts} = 6.43 \text{ volts}$$

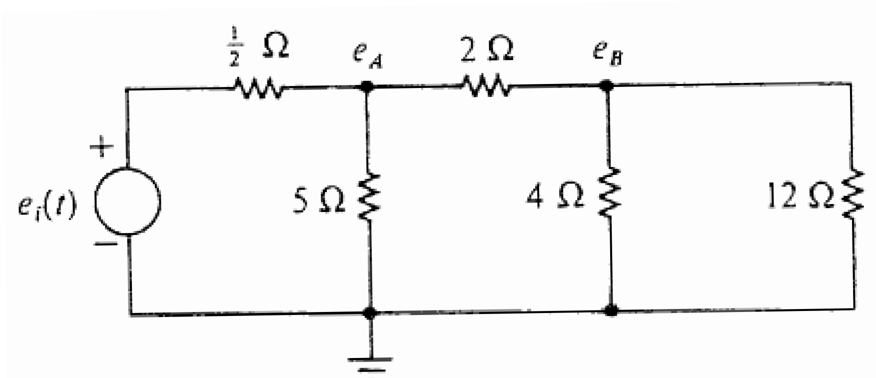
power equation $$P = \frac{e_1^2}{R_3} = \frac{6.43^2 \text{ volt}^2}{220\left(\frac{\text{volt}}{\text{amp}}\right)} = 0.188 \text{ watt}$$

Example 3: Pair-Share Exercise Seven-Resistor Circuit

- The resistive circuit shown consists of a voltage source connected to a combination of seven resistors. The output is voltage e_0. Find the equivalent resistance R_{eq} of the seven-resistor combination and evaluate e_0.



Example 3: Pair-Share Exercise
Seven-Resistor Circuit

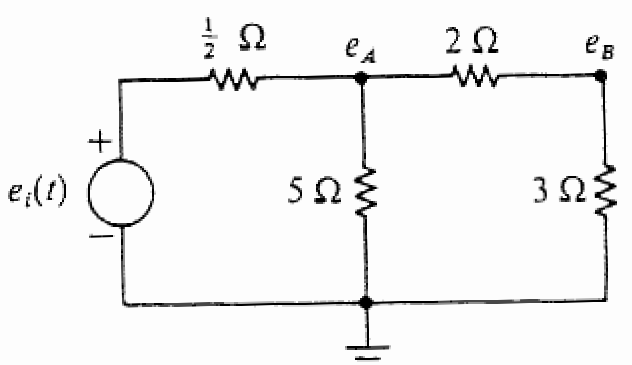


(b)

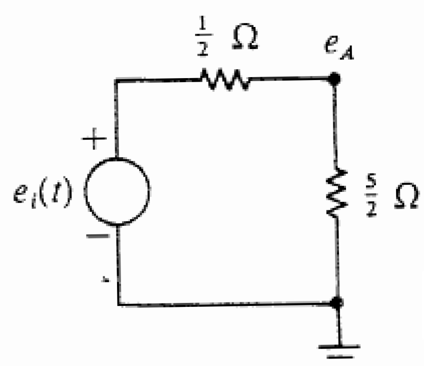
$$R_{eq} = \tfrac{1}{2} + \tfrac{5}{2} = 3\,\Omega$$

Apply voltage divider equation repeatedly

$$e_A = \left(\frac{\frac{5}{2}}{\frac{1}{2} + \frac{5}{2}}\right) e_i(t) = \frac{5}{6} e_i(t)$$



(c)

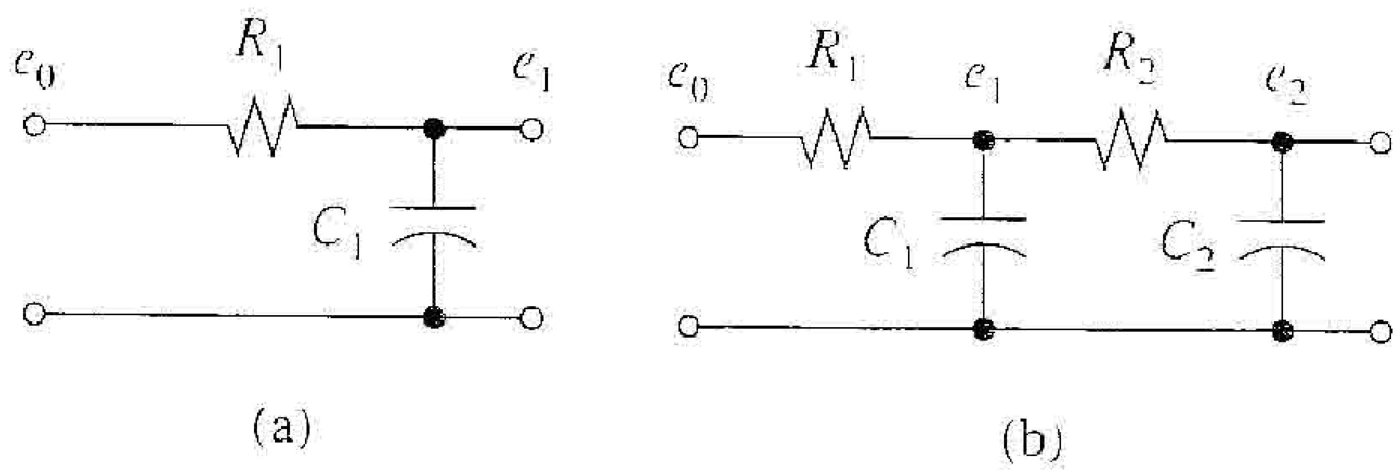


(d)

$$e_B = \left(\frac{3}{2+3}\right) e_A = \frac{1}{2} e_i(t)$$

$$e_o = \left(\frac{2}{2+10}\right) e_B = \frac{1}{12} e_i(t)$$

Example 4: Dual RC Circuit



- Write the modeling equations for circuit (a)
- Derive the differential equation in the form $[\tau D+1]e_1=Ge_0$
- What are the mathematical expressions for time constant τ and gain G?
- For circuit (b), is the differential equation for this circuit a product of two RC's circuit, that is, $[\tau_1 D+1][\tau_2 D+1]e_2=e_0$?

Example 4: Dual RC Circuit

For the circuit of Figure P4.12a, (a) nponent equations are:

$$i_{R_1} = \frac{e_0 - e_1}{R_1}, \quad i_{C_1} = C_1\, De_1 \quad \text{with } e_1(0)$$

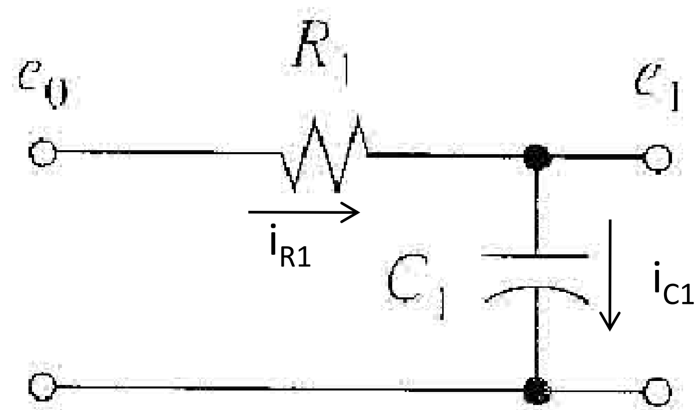
The node equation: $i_{R_1} = i_{C_1}$

Substituting component into node equation:

$$\left(R_1\, C_1\, D + 1\right) e_1 = e_0$$

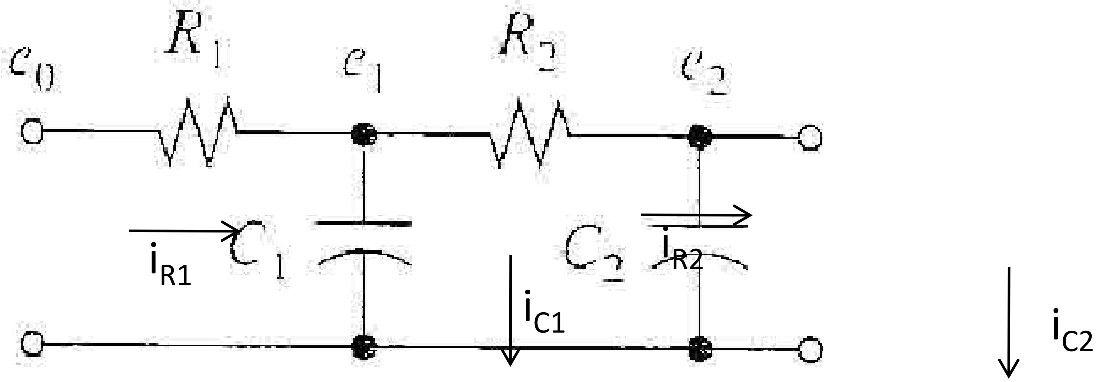
The time constant is $\tau = R_1\, C_1$

The static gain is $G = 1$



(a)

Example 4: Dual RC Circuit



For the circuit of Figure (b), the component equations are:

$$i_{R1} = \frac{e_0 - e_1}{R_1} \qquad i_{C_1} = C_1\ De_1 \qquad \text{with} \quad e_1(0)$$

$$i_{R_2} = \frac{e_1 - e_2}{R_2} \qquad i_{C_2} = C_2\ De_2 \qquad \text{with} \quad e_2(0)$$

The node equations: $i_{R_1} = i_{C_1} + i_{R_2} \qquad i_{R_2} = i_{C_2}$

Example 4: Dual RC Circuit

Substituting component into node equations:

$$\frac{e_0 - e_1}{R_1} = C_1\, De_1 + \frac{e_1 - e_2}{R_2} \qquad \text{or} \qquad \left(R_1\, C_1\, D + 1 + \frac{R_1}{R_2}\right)e_1 = e_0 + e_2$$

$$\frac{e_1 - e_2}{R_2} = C_2\, De_2 \qquad \text{or} \qquad (R_2\, C_2\, D + 1)e_2 = e_1$$

Combining and using $\tau_1 = R_1\, C_1$ and $\tau_2 = R_2\, C_2$:

$$\left(\tau_1\, D + 1 + \frac{R_1}{R_2}\right)(\tau_2\, D + 1)e_2 - e_2 = e_0$$

$$\left[\tau_1 \tau_2 D^2 + \left(\tau_1 + \tau_2 + \frac{R_1}{R_2}\right)D + \frac{R_1}{R_2}\right]e_2 = e_0$$

From the above, we can see that the transfer function is not the product of the two individual transfer functions. This is due to the current being drawn from the first *RC* to drive the second.