Modeling Electrical Systems

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ME584
Agenda

• Basic Electrical Elements
• Passive Circuit Analysis
• Active Circuit Analysis
• Case Study: A Speaker Model

• Active Learning: Pair-share Exercises, Case Study
Basis Electrical Elements
Electrical System

• Composed of resistors, capacitors, inductors, transistors, amplifiers, power supplies
  – Passive circuits: respond to applied voltage or current and do not have any amplifiers
  – Active circuits: made of transistors and/or amplifiers, require active power source to work

• Basic quantities
  – Charge $q$ [coulomb] = $6.24 \times 10^{18}$ electrons
  – Current $i$ [ampere] = $\frac{dq}{dt}$
  – Voltage $e$ [Volt] = $\frac{dw}{dq}$
  – Energy or Work $w$ [joule]
  – Power $p$ [watt] = $e \times i = \frac{dw}{dt}$
# Units and Representations for Common Electrical Quantities

<table>
<thead>
<tr>
<th>Quantity and Symbol</th>
<th>Units</th>
<th>Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Voltage $v$</td>
<td>volt (V)</td>
<td><img src="image" alt="Voltage source" /></td>
</tr>
<tr>
<td>Charge $Q$</td>
<td>coulomb = newton-meter/volt</td>
<td></td>
</tr>
<tr>
<td>Current $i$</td>
<td>ampere (A) = coulomb/second</td>
<td><img src="image" alt="Current source" /></td>
</tr>
<tr>
<td>Flux $\phi$</td>
<td>volt-second</td>
<td></td>
</tr>
<tr>
<td>Resistance $R$</td>
<td>ohm (Ω) = volt/ampere</td>
<td><img src="image" alt="Resistance symbol" /></td>
</tr>
<tr>
<td>Capacitance $C$</td>
<td>farad (F) = coulomb/volt</td>
<td><img src="image" alt="Capacitance symbol" /></td>
</tr>
<tr>
<td>Inductance $L$</td>
<td>henry (H) = volt-second/ampere</td>
<td><img src="image" alt="Inductance symbol" /></td>
</tr>
<tr>
<td>Battery (voltage source plus internal resistance)</td>
<td>–</td>
<td><img src="image" alt="Battery symbol" /></td>
</tr>
<tr>
<td>Ground (zero voltage reference)</td>
<td>–</td>
<td><img src="image" alt="Ground symbol" /></td>
</tr>
</tbody>
</table>
General Model Structure for Mechanical System

\[ M = \int f \, dt \]

\[ x = \int v \, dt \]

Elasticity: \( f = kx \)

Damping: \( f = cv \)

Mass: \( M = mv \)
General Model Structure for Electrical System

\[ \phi = \int v \, dt \]

- Voltage \( v \)
- Flux \( \phi \)
- Capacitance \( C \):
  \[ v = \frac{Q}{C} \]
- Resistance \( R \):
  \[ v = iR \]
- Inductance \( L \):
  \[ \phi = Li \]
- Charge \( Q \):
  \[ Q = \int i \, dt \]
- Current \( i \)
Equivalent Circuit for Spring-Mass-Dashpot Systems

Spring-mass-dashpot system

Equivalent circuit

(All 3 elements share the same displacement) → Series Connection

Connection rule for the equivalent circuit for e → V convention:
- Elements share a common Flow or Displacement → Connected in Series
- Elements share a common Effort → Connected in Parallel
Resistance

- Resistance behavior is between insulator and conductors, allowing a predictable restriction of electron flow

\[ \delta e = Ri \quad \text{where} \quad \delta e = e_1 - e_2 \]

- Power dissipated = \( \delta e i \)

- Resistance \( R = \frac{\rho l}{A} \)
  - A: cross section area of wire
  - l: length of wire
  - \( \rho \): resistivity of material
Capacitance

- Capacitor stores electrons on 2 parallel plates separated by an insulating dielectric material in an electric field

\[ \delta e = \frac{1}{C} \int_{-\infty}^{i} i \, dt \quad \text{or} \quad i = C \frac{d \delta e}{dt} \]

- Energy stored in capacitor

\[ w = \int_{-\infty}^{i} \delta e \, i \, dt = \int_{-\infty}^{i} \delta e \left( C \frac{d \delta e}{dt} \right) \, dt = \int_{0}^{\delta e} C \delta e \, d \delta e = \frac{C \delta e^2}{2} \]

- Capacitance

\[ C = \frac{\epsilon A}{d} \]

- A: area of plates
- D: spacing between plates
- \( \epsilon \): permittivity of the dielectric
Inductance

• Inductance relates voltage induced to time rate of change of magnetic field

• Faraday’s law:

\[ e_i = \frac{d\phi}{dt} \quad \text{where } \phi \text{ is the magnetic flux, } \phi = Li \]

\[ \delta e = L \frac{di}{dt} \quad \text{where } \delta e = e_1 - e_2 \]

• Energy stored

\[ w = \int_{-\infty}^{t} \delta e \, i \, dt = \int_{-\infty}^{t} \left( L \frac{di}{dt} \right) i \, dt = \int_{0}^{i} Li \, di = \frac{Li^2}{2} \]

• Inductance:

\[ L = \frac{\mu_m n^2}{\ell} \cdot A \]

where \( A = \) wire cross section area, \( l = \) wire length, \( n = \) number of turns, \( \mu_m = \) permeability of magnetic circuit
Impedance

• Impedance $Z$: instantaneous ratio of voltage difference to current

$$Z = \frac{\delta e}{i}$$

• Impedance of common circuit elements
  – Resistive: $Z_r = R$ (not dynamic)
  – Capacitive: $Z_c = 1/CD$
  – Inductive: $Z_L = LD$

Where $D =$ differential operator $d/dt$,

$1/D =$ integrator operator
## Ideal and Non-Ideal Sources

<table>
<thead>
<tr>
<th></th>
<th>Voltage Source</th>
<th>Current Source</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Ideal</strong></td>
<td><img src="image" alt="Ideal Voltage Source" /></td>
<td><img src="image" alt="Ideal Current Source" /></td>
</tr>
<tr>
<td><strong>Non-Ideal</strong></td>
<td><img src="image" alt="Non-Ideal Voltage Source" /></td>
<td><img src="image" alt="Non-Ideal Current Source" /></td>
</tr>
<tr>
<td><strong>Battery</strong></td>
<td><img src="image" alt="Battery" /></td>
<td></td>
</tr>
</tbody>
</table>
Open and Short Circuits

• An open circuit is any element through which current cannot flow

• A short circuit is any element across which there is no voltage
Series and Parallel Impedance Combinations

**Series**

<table>
<thead>
<tr>
<th>R</th>
<th>$e_1$</th>
<th>$Z_{eq}$</th>
<th>$e_2$</th>
</tr>
</thead>
</table>

Equivalent

$R_{eq} = R_1 + R_2$


<table>
<thead>
<tr>
<th>C</th>
<th>$e_1$</th>
<th>$Z_{eq}$</th>
<th>$e_2$</th>
</tr>
</thead>
</table>

Equivalent

$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$


<table>
<thead>
<tr>
<th>L</th>
<th>$e_1$</th>
<th>$Z_{eq}$</th>
<th>$e_2$</th>
</tr>
</thead>
</table>

Equivalent

$L_{eq} = L_1 + L_2$


**Parallel**

<table>
<thead>
<tr>
<th>R</th>
<th>$e_1$</th>
<th>$Z_1$</th>
<th>$Z_2$</th>
<th>$e_2$</th>
</tr>
</thead>
</table>


<table>
<thead>
<tr>
<th>C</th>
<th>$e_1$</th>
<th>$Z_1$</th>
<th>$Z_2$</th>
<th>$e_2$</th>
</tr>
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</table>


<table>
<thead>
<tr>
<th>L</th>
<th>$e_1$</th>
<th>$Z_1$</th>
<th>$Z_2$</th>
<th>$e_2$</th>
</tr>
</thead>
</table>


$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$

$C_{eq} = C_1 + C_2$

$L_{eq} = \frac{1}{L_1} + \frac{1}{L_2}$
Laws for Passive Circuit Analysis

- **Kirchhoff’s Current Law (KCL)**
  - The sum of all currents (flow) entering a node is zero
- **Kirchhoff’s Voltage Law (KVL)**
  - The oriented sum of all voltages (efforts) around any closed loop is zero
Techniques for Passive Circuit Analysis for Classical Deriving Differential Equations

1. Draw circuit schematic and label components (e.g., $R_1$, $R_2$, $C_1$, $L_1$...)
2. Assign voltage at each node (e.g., $e_1$, $e_2$)
3. Assign current in each component (e.g., $i_1$, $i_2$, ..) and show positive current direction with arrows
4. Write equation for current for each component (e.g., $i_{R1} = (e_1-e_2)/R_1$ or $i_{C1} = CDe_1$)
5. Write node equations for each significant node (not connected to voltage or current source)
6. Substitute component equations into node equations and reduce results to a single differential equation with output and input variables
Example 1: Voltage Divider

Evaluate $e_1$

\[ i_1 = \frac{e_0 - e_1}{R_1} \]
\[ i_2 = \frac{e_1 - 0}{R_2} \]
\[ i_1 - i_2 = 0 \]

\[ \frac{e_0 - e_1}{R_1} - \frac{e_1}{R_2} = 0 \]
\[ e_1 = \frac{R_2}{R_1 + R_2} e_0 \]
\[ e_1 = \frac{1}{1 + \frac{R_1}{R_2}} e_0 \]
Example 2: Resistor Circuit

- Calculate the amount of power dissipated in resistor $R_3$ in the circuit shown below.

- Solution:

  - Resistors in parallel
    
    $$R_{eq} = \frac{R_2 R_3}{R_2 + R_3} = \frac{1000 \times 220}{1000 + 220} = 180.3 \, \Omega$$

  - Voltage divider equation
    
    $$e_1 = \frac{R_{eq}}{R_1 + R_{eq}} e_0 = \frac{180.3}{100 + 180.3} \times 10 \text{ volts} = 6.43 \text{ volts}$$

  - Power equation
    
    $$P = \frac{e_1^2}{R_3} = \frac{6.43^2 \text{ volt}^2}{220 \left( \frac{\text{volt}}{\text{amp}} \right)} = 0.188 \text{ watt}$$
Example 3: Pair-Share Exercise
Seven-Resistor Circuit

- The resistive circuit shown consists of a voltage source connected to a combination of seven resistors. The output is voltage $e_0$. Find the equivalent resistance $R_{eq}$ of the seven-resistor combination and evaluate $e_0$. 

[Diagram of the circuit]
Example 3: Pair-Share Exercise
Seven-Resistor Circuit

Apply voltage divider equation repeatedly

\[ R_{eq} = \frac{1}{2} + \frac{5}{2} = 3 \, \Omega \]

\[ e_A = \left( \frac{\frac{5}{2}}{\frac{1}{2} + \frac{5}{2}} \right) e_i(t) = \frac{5}{6} e_i(t) \]

\[ e_B = \left( \frac{\frac{3}{2+3}}{\frac{1}{2} + \frac{3}{2+3}} \right) e_A = \frac{1}{2} e_i(t) \]

\[ e_o = \left( \frac{\frac{2}{2+10}}{\frac{1}{2} + \frac{2}{2+10}} \right) e_B = \frac{1}{12} e_i(t) \]
Example 4: Dual RC Circuit

- Write the modeling equations for circuit (a)
- Derive the differential equation in the form $[\tau D+1]e_1 = Ge_0$
- What are the mathematical expressions for time constant $\tau$ and gain $G$?
- For circuit (b), is the differential equation for this circuit a product of two RC’s circuit, that is, $[\tau_1 D+1][\tau_2 D+1]e_2 = e_0$?
Example 4: Dual RC Circuit

For the circuit of Figure P4.12a, the component equations are:

\[ i_{R_1} = \frac{e_0 - e_1}{R_1}, \quad i_{C_1} = C_1 De_1 \quad \text{with} \quad e_1(0) \]

The node equation:

\[ i_{R_1} = i_{C_1} \]

Substituting component into node equation:

\[ (R_1 C_1 D + 1) e_1 = e_0 \]

The time constant is

\[ \tau = R_1 C_1 \]

The static gain is

\[ G = 1 \]
Example 4: Dual RC Circuit

For the circuit of Figure (b), the component equations are:

\[ i_{R1} = \frac{e_0 - e_1}{R_1} \quad \quad i_{C1} = C_1 \, De_1 \quad \text{with} \quad e_1(0) \]

\[ i_{R2} = \frac{e_1 - e_2}{R_2} \quad \quad i_{C2} = C_2 \, De_2 \quad \text{with} \quad e_2(0) \]

The node equations: \[ i_{R1} = i_{C1} + i_{R2} \quad \quad i_{R2} = i_{C2} \]
Example 4: Dual RC Circuit

Substituting component into node equations:

\[
\frac{e_0 - e_1}{R_1} = C_1 D e_1 + \frac{e_1 - e_2}{R_2} \quad \text{or} \quad \left( R_1 C_1 D + 1 + \frac{R_1}{R_2} \right) e_1 = e_0 + e_2
\]

\[
\frac{e_1 - e_2}{R_2} = C_2 D e_2 \quad \text{or} \quad \left( R_2 C_2 D + 1 \right) e_2 = e_1
\]

Combining and using \( \tau_1 = R_1 C_1 \) and \( \tau_2 = R_2 C_2 \):

\[
\left( \tau_1 D + 1 + \frac{R_1}{R_2} \right) \left( \tau_2 D + 1 \right) e_2 - e_2 = e_0
\]

\[
\left[ \tau_1 \tau_2 D^2 + \left( \tau_1 + \tau_2 + \frac{R_1}{R_2} \right) D + \frac{R_1}{R_2} \right] e_2 = e_0
\]

From the above, we can see that the transfer function is not the product of the two individual transfer functions. This is due to the current being drawn from the first \( RC \) to drive the second.