Lagrange's Equation

Lagrange's Method

- Newton's method of developing equations of motion requires taking elements apart

- When forces at interconnections are not of primary interest, more advantageous to derive equations of motion by considering energies in the system

- Lagrange's equations:
  - Indirect approach that can be applied for other types of systems (other than mechanical)
  - Based on calculus of variations – finding extremums of quantifies expressible as integrals

Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial U}{\partial q_i} = Q_i$$

$where$:

q_i : independent coordinates necessary to

　　describe system's motion at any instant

Q_i : corresponding loading in each coordinate

$U = f_1(q_i)$: potential energy in terms of coordinates

$T = f_2(\dot{q}_i^2)$: kinetic energy in terms of system masses,

　　　　　　　mass inertias, linear/angular velocities

$R = f_3(\dot{q}_i^2)$: energy dissipation due to viscous friction

Deriving Equations of Motion via Lagrange's Method

1. Select a complete and independent set of coordinates q_i's

2. Identify loading Q_i in each coordinate

3. Derive T, U, R

4. Substitute the results from 1,2, and 3 into the Lagrange's equation.

Example 11: Spring-Mass-Damper System

Independent coordinate: q = x

$$U = \frac{1}{2} kx^2 \qquad R = \frac{1}{2} b\dot{x}^2$$
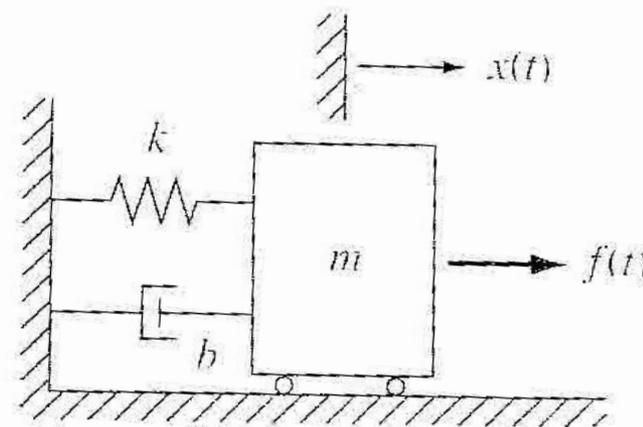
$$T = \frac{1}{2} m\dot{x}^2 \qquad Q = f(t)$$

$$\frac{\partial R}{\partial \dot{x}} = b\dot{x}$$
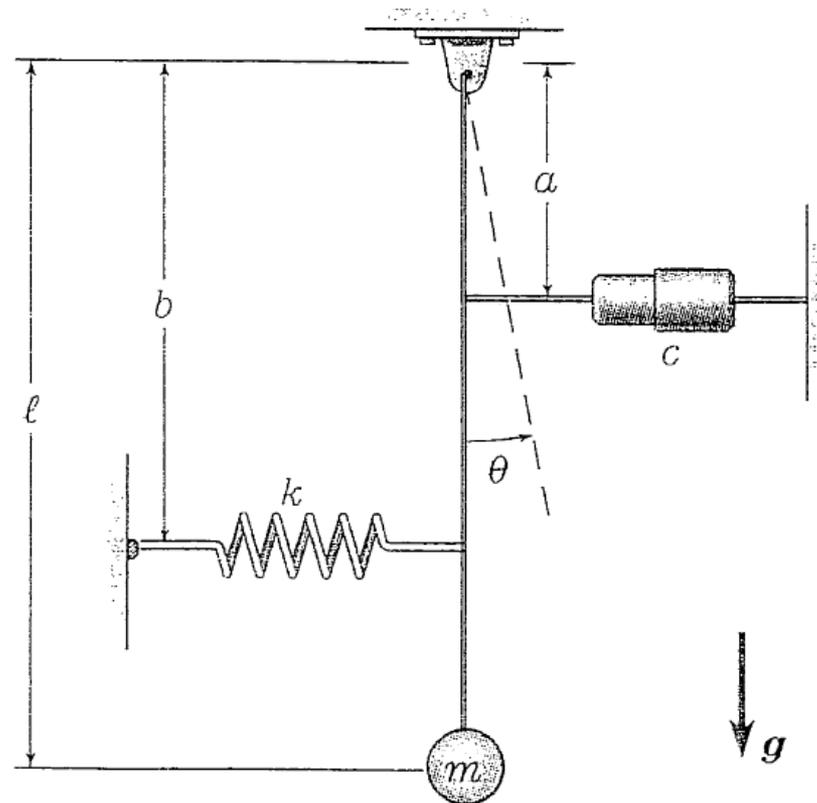
$$\frac{\partial U}{\partial x} = kx$$

Substitute into Lagrange's equation:

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

Example 12: Pair-Share: Restrained Plane Pendulum

- A plane pendulum (length l and mass m), restrained by a linear spring of spring constant k and a linear dashpot of dashpot constant c, is shown on the right. The upper end of the rigid massless link is supported by a frictionless joint. Derive the equation of motion.

Example 12: Pair-Share: Restrained Plane Pendulum

$$q = \theta, \quad Q = 0,$$

$$R = \frac{1}{2}c(a\dot{\theta})^2$$
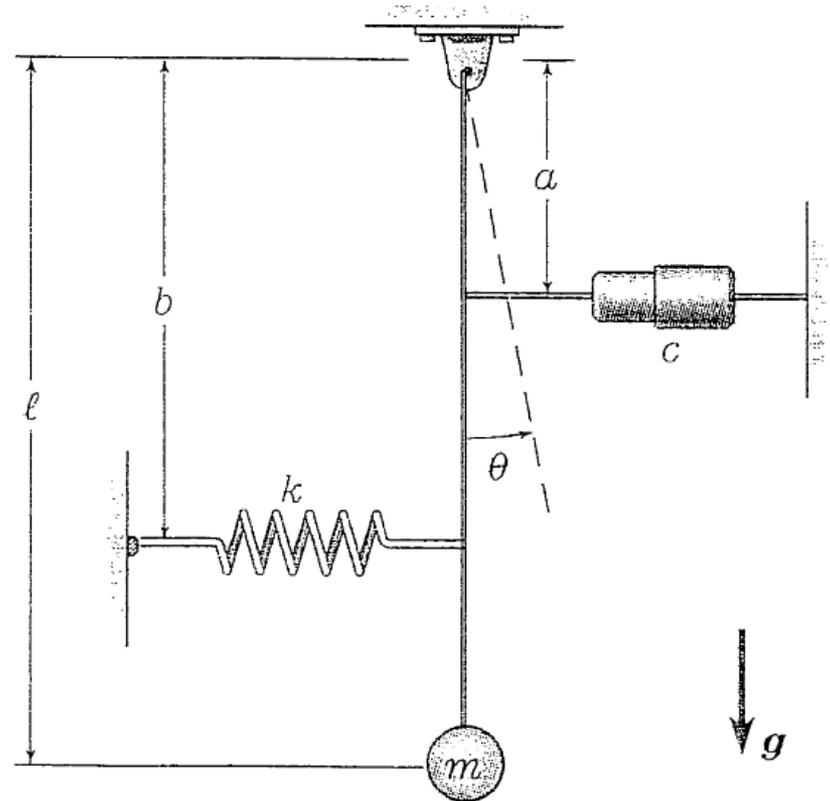
$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot{\theta})^2$$

$$U = mgl - mgl\cos\theta + \frac{1}{2}k(b\theta)^2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial R}{\partial \dot{\theta}} + \frac{\partial U}{\partial \theta} = Q$$

$$ml^2\ddot{\theta} + ca^2\dot{\theta} + mgl\sin\theta + kb^2\theta = 0$$

$Linearize \sin\theta \approx \theta \ for \ small \ \theta$

$$ml^2\ddot{\theta} + ca^2\dot{\theta} + mgl\theta + kb^2\theta = 0$$

Example 13: Bead on a Spinning Wire Hoop

- One degree of freedom, q_1=angle θ as an independent coordinate
- Velocity of bead: $R\dot{\theta}$. ; Velocity of hoop: $\rho\omega = R\sin\theta\,\omega$
- Kinetic energy: $$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left[(R\dot{\theta})^2 + (R\omega\sin\theta)^2\right]$$

- Potential energy relative to its position at the bottom of the hoop (when the hoop is not rotating and $\theta = 0$), is
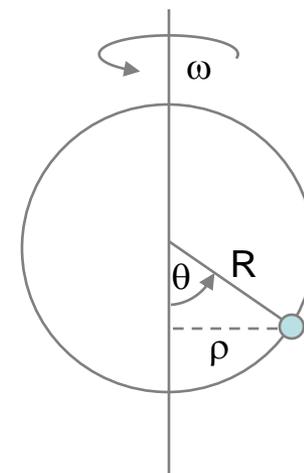$$U = mgR - mgR\cos\theta$$
- R = 0, Q = 0
- Substitute into Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} + \frac{\partial R}{\partial\dot{\theta}} + \frac{\partial U}{\partial\theta} = Q$$

$$mR^2\ddot{\theta} - mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta = 0$$

- Solving for the angular acceleration: $\ddot{\theta} = \left(\omega^2\cos\theta - \frac{g}{R}\right)\sin\theta.$

Example 14: Pair-Share: Copying machine

- Use Lagrange's equation to derive the equations of motion for the copying machine example, assuming potential energy due to gravity is negligible.

$q_1=y$, $q_2= \theta$

Since the bar mass is negligible, the kinetic energy is:
$$T = \frac{1}{2}m_2 \dot{y}^2$$

The potential energy is:
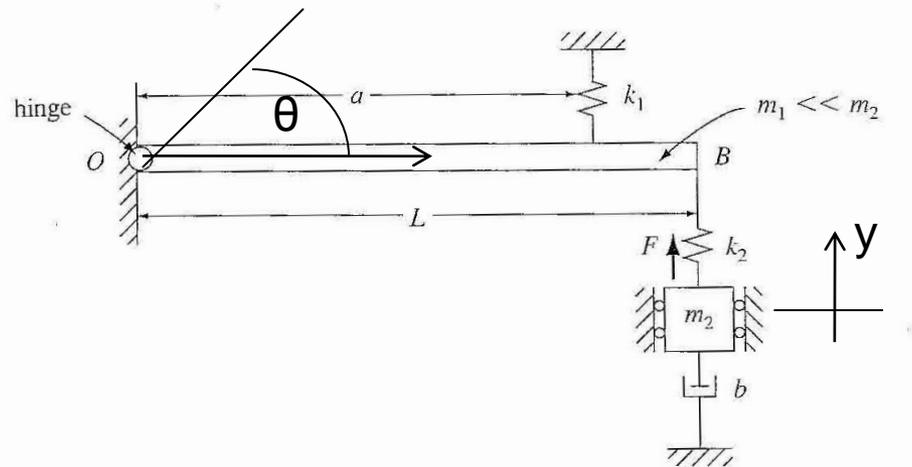$$U = \frac{1}{2}k_2(L\theta - y)^2 + \frac{1}{2}k_1(a\theta)^2$$

While the dissipation function is given by:
$$R = \frac{1}{2}b(\dot{y})^2$$

Lagrange's equation:
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial U}{\partial q_i} = Q$$

$$Q_1 = F, \quad Q_2 = 0$$

Example 14: Pair-Share: Copying machine

The required derivatives are:

$$\frac{\partial T}{\partial \dot{y}} = m_2 \, \dot{y} \qquad\qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = m_2 \, \ddot{y} \qquad\qquad \frac{\partial T}{\partial \dot{\theta}} = 0$$

$$\frac{\partial U}{\partial y} = -k_2 (L\theta - y) \qquad\qquad \frac{\partial U}{\partial \theta} = k_2 L(L\theta - y) + k_1 a^2 \theta$$

$$\frac{\partial R}{\partial \dot{y}} = b\dot{y}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial U}{\partial \theta} = 0 \qquad\qquad \text{But } \frac{\partial T}{\partial \dot{\theta}} = 0 \quad \text{giving}$$

$$k_2 L^2 \theta - k_2 L y + k_1 a^2 \theta = 0 \quad \text{or} \qquad (k_2 L^2 + k_1 a^2)\theta = k_2 L y$$

Thus

$$\theta = \frac{k_2 L}{(k_2 L^2 + k_1 a^2)} y$$

Substitution gives:

$$m_2 \ddot{y} + b\dot{y} - k_2 (L\theta - y) = F$$

$$m_2 \ddot{y} + b\dot{y} + k_2 y - k_2 L\frac{k_2 L}{(k_2 L^2 + k_1 a^2)} y = F$$

Example 15: Mass Spring Dashpot Subsystem in Falling Container

- A mass spring dashpot subsystem in a falling container of mass m_1 is shown. The system is subject to constraints (not shown) that confine its motion to the vertical direction only. The mass m_2, linear spring of undeformed length l_0 and spring constant k, and the linear dashpot of dashpot constant c of the internal subsystem are also shown.

Container

- Derive equation(s) of motion for the system using
  - x_1 and x_2 as independent coordinates
  - y_1 and y_2 as independent coordinates
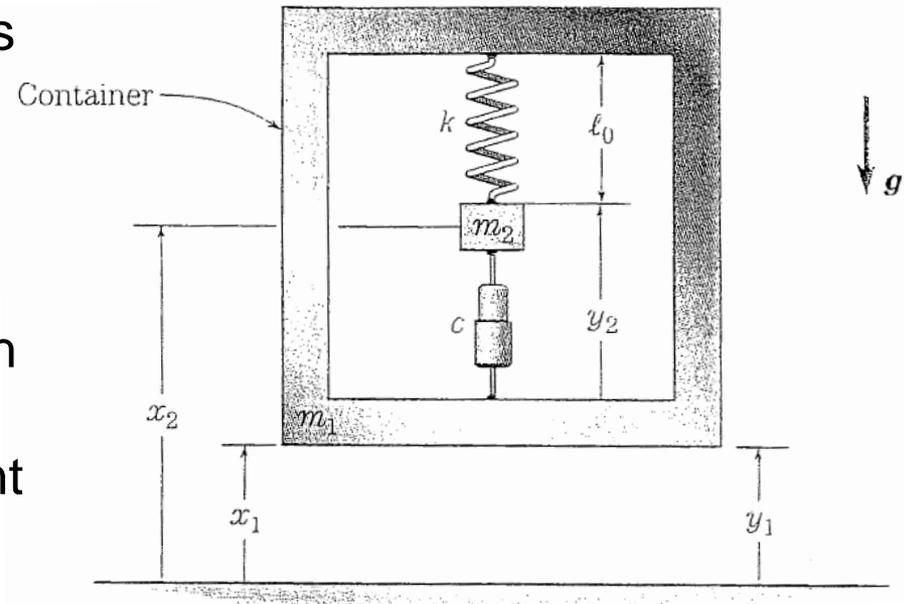
Example 15: Mass Spring Dashpot Subsystem in Falling Container

Let $q_1 = x_1, q_2 = x_2$ *be positions of* m_1 *and* m_2 *with respect to horizontal ground.*

$Q_{1,2} = 0$

$T = \dfrac{1}{2} m_1 \dot{x}_1^2 + \dfrac{1}{2} m_2 \dot{x}_2^2$

$U = \dfrac{1}{2} k (x_1 - x_2)^2 + m_1 g x_1 + m_2 g x_2$
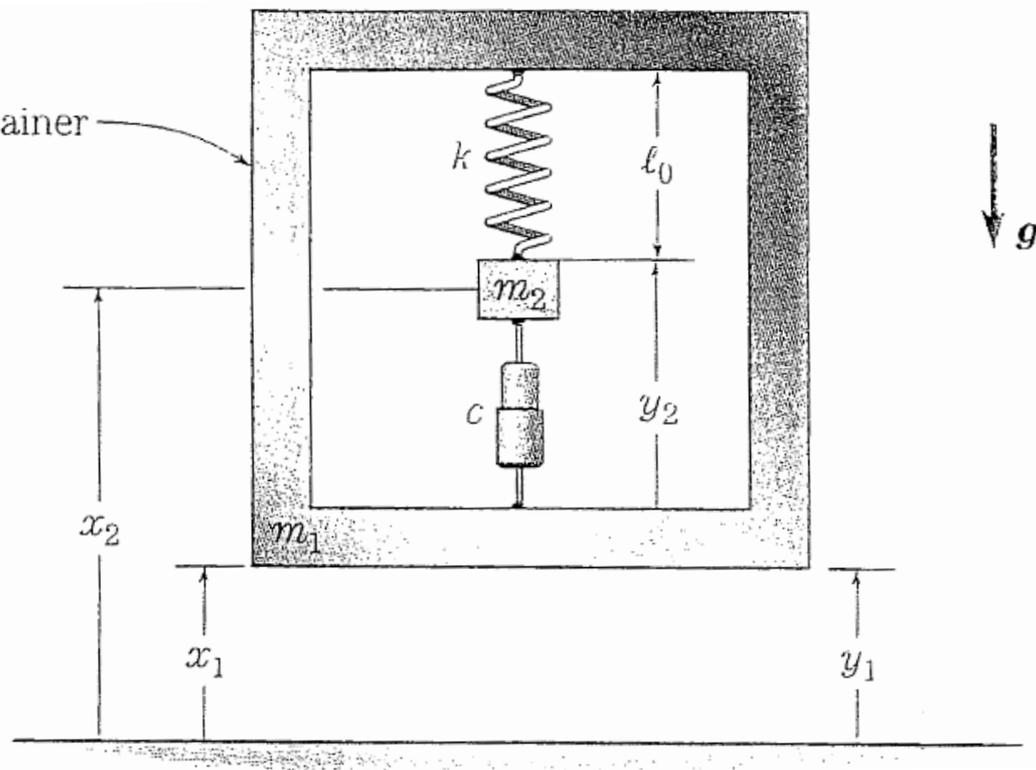
$R = \dfrac{1}{2} c (\dot{x}_1 - \dot{x}_2)^2$

Substitute into Lagrange's Equation :

$\dfrac{d}{dt} \left(\dfrac{\partial T}{\partial \dot{x}_1} \right) - \dfrac{\partial T}{\partial x_1} + \dfrac{\partial R}{\partial \dot{x}_1} + \dfrac{\partial U}{\partial x_1} = Q_1$

$m_1 \ddot{x}_1 + c (\dot{x}_1 - \dot{x}_2) + k (x_1 - x_2) + m_1 g = 0$

$\dfrac{d}{dt} \left(\dfrac{\partial T}{\partial \dot{x}_2} \right) - \dfrac{\partial T}{\partial x_2} + \dfrac{\partial R}{\partial \dot{x}_2} + \dfrac{\partial U}{\partial x_2} = Q_2$

$m_1 \ddot{x}_2 + c (\dot{x}_2 - \dot{x}_1) + k (x_2 - x_1) + m_2 g = 0$

Container

k ℓ_0

m_2

c y_2

x_2

m_1

x_1 y_1

g

Example 15: Mass Spring Dashpot Subsystem in Falling Container

Let $q_1 = y_1$ *be inertial position of* m_1 *with respect to horizontal ground*

  $q_2 = y_2$ *be position of* m_2 *relative to* m_1 *and from unstretched length of spring*

$Q_{1,2} = 0$

$$T = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2(\dot{y}_1 + \dot{y}_2)^2$$

$$U = \frac{1}{2}ky_2^2 + m_1 g y_1 + m_2 g(y_1 + y_2)$$
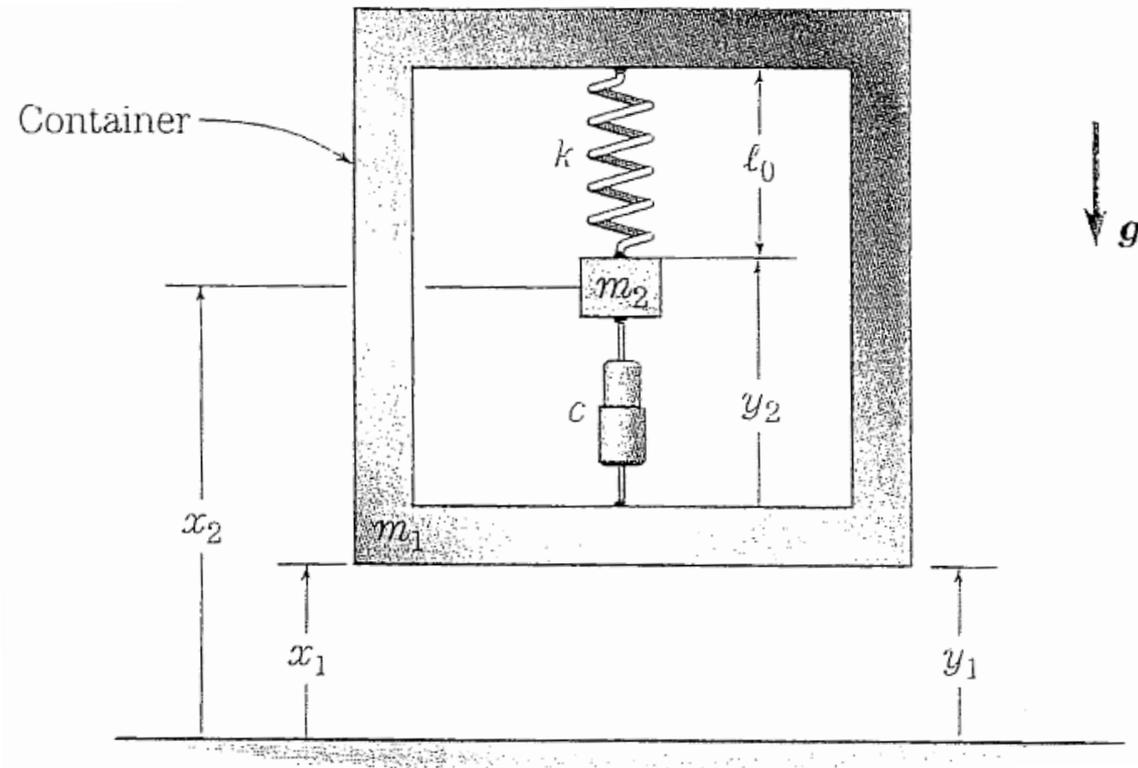
$$R = \frac{1}{2}c\dot{y}_2^2$$

Substitute into Lagrange's Equation :

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{y}_1}) - \frac{\partial T}{\partial y_1} + \frac{\partial R}{\partial \dot{y}_1} + \frac{\partial U}{\partial y_1} = Q_1$$

$$(m_1 + m_2)\ddot{y}_1 + m_2\ddot{y}_2 + m_1 g + m_2 g = 0$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{y}_2}) - \frac{\partial T}{\partial y_2} + \frac{\partial R}{\partial \dot{y}_2} + \frac{\partial U}{\partial y_2} = Q_2$$
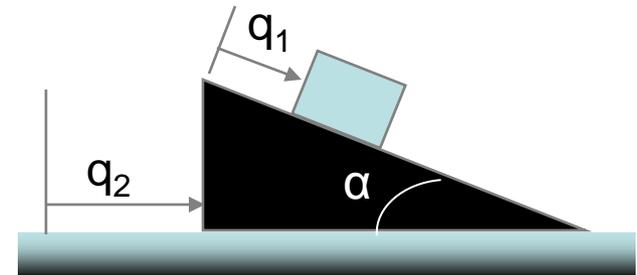
$$m_2\ddot{y}_1 + m_2\ddot{y}_2 + c\dot{y}_2 + ky_2 + m_2 g = 0$$

Example 16: A Block Sliding on a Wedge

- Block mass m sliding down a wedge mass M
- Independent coordinates, q_1 and q_2, are shown, q_1 is along the plane and is measured relative to the (moving) wedge.
- Velocity of the wedge is $\dot{q}_2$, but velocity of the block has components from both q_1 and q_2:

$$v_{block}^2 = v_x^2 + v_y^2 = \left(\dot{q}_2 + \dot{q}_1 \cos\alpha\right)^2 + \left(\dot{q}_1 \sin\alpha\right)^2$$

- Total kinetic energy is $\quad T = \dfrac{1}{2} M \dot{q}_2^2 + \dfrac{1}{2} m \left[\left(\dot{q}_2 + \dot{q}_1 \cos\alpha\right)^2 + \left(\dot{q}_1 \sin\alpha\right)^2\right]$

- The potential energy of the wedge can be taken as zero, and the block is

$$U = -mgq_1 \sin\alpha$$

- Substitute into Lagrange's equations and differentiate wrt to q_1 and q_2

$$\ddot{q}_1 = \frac{g \sin\alpha}{1 - \dfrac{m}{M+m}\cos^2\alpha}. \qquad\qquad \ddot{q}_2 = -\frac{m}{M+m}\ddot{q}_1 \cos\alpha.$$

Example 17: Pair-Share: Mass Pendulum Dynamic System

- A simple plane pendulum of mass m_0 and length l is suspended from a cart of mass m as sketched in the figure. The motion of the cart is restrained by a spring of spring constant k and a dashpot constant c; and the angle of the pendulum is restrained by a torsional spring of spring constant k, and a torsional dashpot of dashpot constant c_t. Note that the constitutive relations for the torsional spring and torsional dashpot are linear expressions $\tau = k_t\theta$ and $\tau = c_t\dot{\theta}$, respectively, where τ is the torque and θ is the change in the angle across the element from its undeformed configuration.
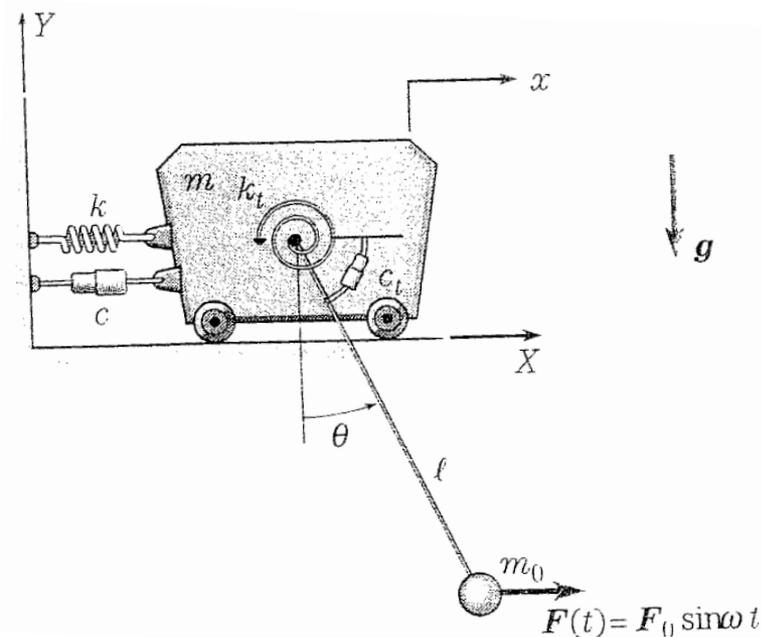
- The torsional spring is undeformed when the pendulum is in its downward-hanging equilibrium position. Also, m_0 is acted upon by a known dynamic force $F_0\sin\omega t$, which remains horizontal, regardless of the angle θ and the motion of the cart. The system is constrained to remain in the plane of the sketch and the cart remains on the bed throughout its motion. Derive the equation(s) of motion for the system.

Example 16: Pair-Share:
Mass Pendulum Dynamic System

Two independent coordinates : $q_1 = x, q_2 = \theta$

$Q_1 = F_0 \sin \omega t \quad and \quad Q_2 = (F_0 \sin \omega t)l \cos \theta$

$R_x = \dfrac{1}{2}c\dot{x}^2 \quad and \quad R_\theta = \dfrac{1}{2}c_t\dot{\theta}^2$

$U = \dfrac{1}{2}kx^2 + \dfrac{1}{2}k_t\theta^2 + m_0 gl(1 - \cos \theta)$
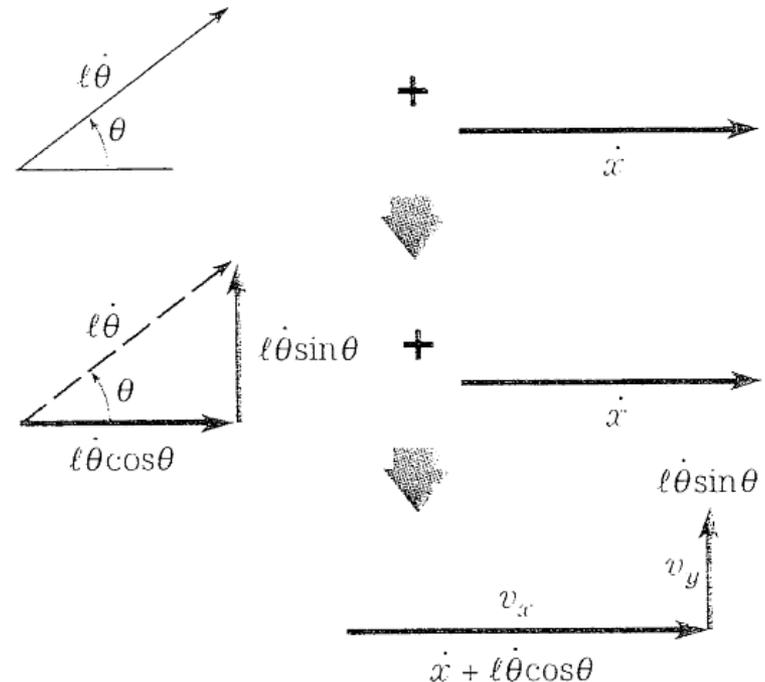
$T = T_{cart} + T_{pendulum}$

$T_{cart} = \dfrac{1}{2}m\dot{x}^2$

$T_{pendulum} = \dfrac{1}{2}m_0\left((\dot{x} + l\dot{\theta}\cos \theta)^2 + (l\dot{\theta}\sin \theta)^2\right) = \dfrac{1}{2}m_0\left(\dot{x}^2 + (l\dot{\theta})^2 + 2\dot{x}l\dot{\theta}\cos \theta\right)$

Substitute into Lagrange's Equations :

$(m + m_0)\ddot{x} + c\dot{x} + kx + m_0 l(\ddot{\theta}\cos \theta - \dot{\theta}^2 \sin \theta) = F_0 \sin \omega t$

$m_0 l\ddot{\theta} + c_t\dot{\theta} + k_t\theta + m_0 gl \sin \theta + m_0\ddot{x}l \cos \theta = (F_0 \sin \omega t)l \cos \theta$

Comparison of Newton's and Lagrange's Methods

Newton's (Direct Approach)	Lagrange's (Indirect Approach)
Accelerations required	Velocities required
Generally vectors required	Generally scalars required
Free-body Diagrams useful	Free-body diagrams not useful
All forces considered	Workless forces (constraints) forces not considered
All forces handled via same expression	Conservative and non-conservative forces handled separately
Intermediate forces more readily available	Intermediate forces less readily available

Case Study:
Feasibility Study of a Mobile Robot Design

Preliminary Design Model

- Read Handout 1

- Develop a simplified model of the base and the arm by neglecting the reaction forces that occur at the pivot

- When do the reaction forces become significant?

=> at high accelerations and speeds of the base and the arm

Preliminary Design Model

We will develop simplified models of the base and the arm by neglecting the reaction forces that occur at the pivot. These reaction forces become more significant at higher accelerations and speeds of the base and arm. Thus the following models assume that these speeds and accelerations are small. We will later develop more complex models that account for these reactions.

A simple model of the base is shown in Figure 2.6-2a. The base motor force f must accelerate the base mass M, the arm mass m_1, and the hand mass m_2. Thus, applying Newton's law, we obtain

$$(M + m_1 + m_2)\ddot{x} = 140\ddot{x} = f \tag{2.6-1}$$

A simple model of the arm-hand combination is shown in Figure 2.6-2b, where m is the combined arm-hand mass: $m = m_1 + m_2 = 30 + 10 = 40$ kg. The center of
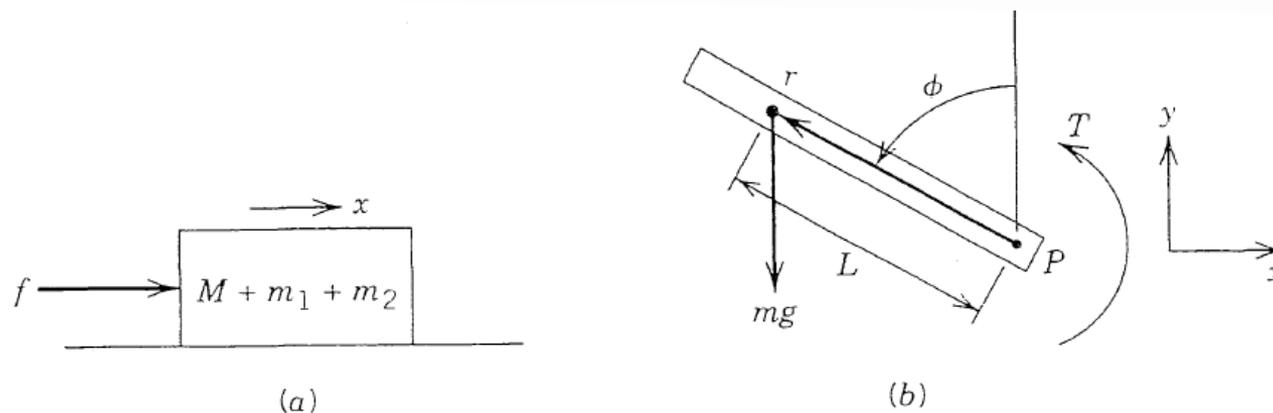

(a)   (b)

Figure 2.6-2 A robot model that neglects reaction forces. (a) Model of the base. (b) Model of the arm.

Preliminary Design Model

mass is a distance L from the pivot point P. Thus

$$L = \frac{m_1 D_1 + m_2(r_2 + D_2)}{m_1 + m_2} = \frac{7}{8} \text{ m}$$

The moment of inertia I_P about the pivot is

$$I_P = m_1 D_1^2 + m_2 (D_2 + r_2)^2 = 30(0.5)^2 + 10(2 + 0)^2 = 47.5 \text{ kg-m}^2$$

Note that the pivot is accelerating horizontally with an acceleration $\ddot{x}$. From Figure 2.6-2b and the moment equation (2.5-4), we find that the arm's equation of motion is

$$I_P \ddot{\phi} + m r_x a_{Py} - m r_y a_{Px} = M_P = T + mgL \sin \phi$$

where $r_x = -L \sin \phi$, $r_y = L \cos \phi$, $a_{Px} = \ddot{x}$, $a_{Py} = 0$. Thus the equation of motion for the arm is

$$I_P \ddot{\phi} - mL\ddot{x} \cos \phi = T + mgL \sin \phi \qquad (2.6\text{-}2)$$

or

$$47.5 \ddot{\phi} - 35 \ddot{x} \cos \phi = T + 343.4 \sin \phi \qquad (2.6\text{-}3)$$

Motion Profiles

- Read Handout 2
- Plot the motor force versus time, and motor torque versus time, and determine whether the motor is powerful enough

Motion Profiles

From (2.6-4) we see that the maximum magnitude of the base acceleration occurs at $t = 0$ and at $t = 4$, and is $A = 1.875$ m/s². Using model (2.6-1) with $\ddot{x} = A$, we find that the maximum motor force f required is $f = 140A = 262.5$ N, which is less than the motor's maximum of 300 N. Thus the base motor appears to be sufficient for the application. The motor force as a function of time is obtained by substituting $\ddot{x}$ from (2.6-4) into (2.6-1) to obtain $f = 140(1.875)(1 - t/2)$. The curve is labeled f_1 in Figure 2.6-4. The curve labeled f_2 will be discussed shortly.

We can plot the motor torque T versus time using (2.6-4), (2.6-7), and (2.6-8) in the arm model (2.6-3). The plot is shown in Figure 2.6-4. It shows that the maximum required torque magnitude is 371 N-m at $t = 0$. Because the torque magnitude is always less than the motor's maximum of 400 N m, the arm motor appears to be powerful enough.

Figure 2.6-4 Predicted base force f and arm torque T as functions of time.
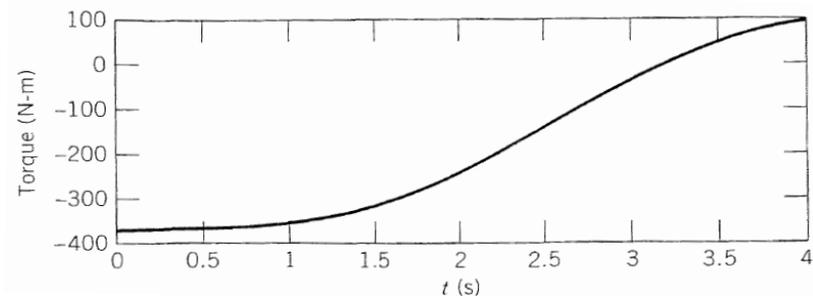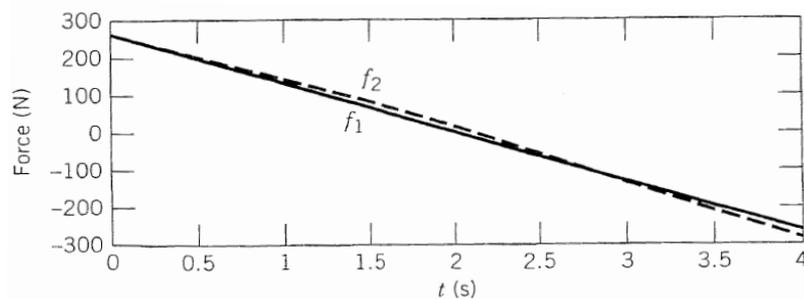
A More Detailed Model

- The simplified model appears feasible
- Develop a detailed model, including the reaction forces to see whether they significantly change the results
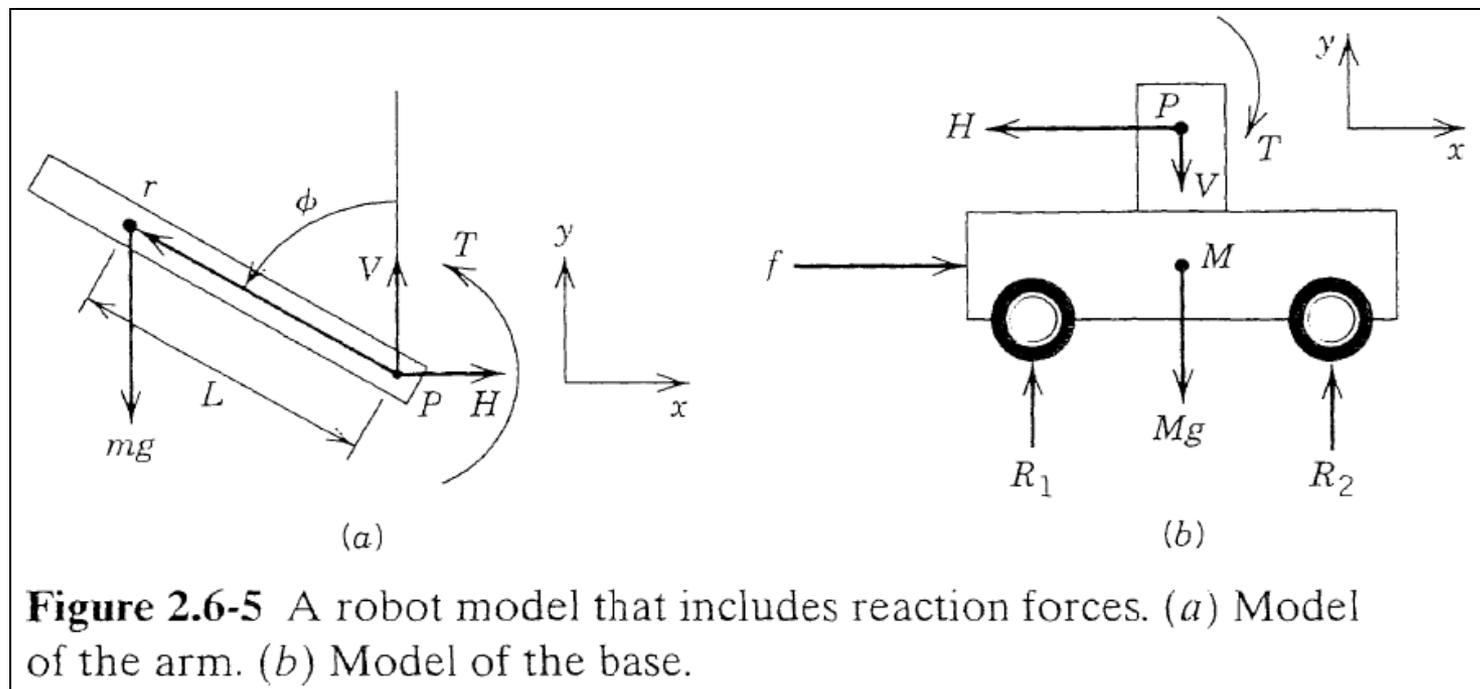
A More Detailed Model

Figure 2.6-5 A robot model that includes reaction forces. (*a*) Model of the arm. (*b*) Model of the base.

The free body diagrams are shown in Figure 2.6-5. First consider the arm. The vertical and horizontal components of the arm's mass center are $L \cos \phi$ and $x + L \sin \phi$, respectively. For the horizontal direction Newton's law gives

$$m\frac{d^2}{dt^2}(x - L \sin \phi) = H \tag{2.6-9}$$

where H is the horizontal component of the reaction force at the pivot. The moment equation (2.6-4) about the arm's pivot point P gives:

$$I_P\ddot{\phi} - mL\ddot{x} \cos \phi = T + mgL \sin \phi \tag{2.6-10}$$

A More Detailed Model

where I_P is the arm's moment of inertia about the pivot point.

Now consider the base. Newton's law in the horizontal direction gives

$$M\ddot{x} = f - H \tag{2.6-11}$$

Because we are assuming that the base does not rotate or move vertically, we need not consider the moments and vertical forces on the base, unless we need to compute the reaction forces V, R_1, and R_2.

This completes the equations of motion for the system. These equations can be put into a more useful form, as follows. Evaluate the derivative in (2.6-9) to obtain

$$m\ddot{x} - mL\frac{d}{dt}(\cos\phi\,\dot{\phi}) = m\ddot{x} - mL(-\sin\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{\phi}) = H \tag{2.6-12}$$

Solve (2.6-11) for the reaction force H: $H = f - M\ddot{x}$. Substitute this into (2.6-12) and collect terms to obtain:

$$(m + M)\ddot{x} - mL(\cos\phi\,\ddot{\phi} - \sin\phi\,\dot{\phi}^2) = f \tag{2.6-13}$$

A More Detailed Model

The equations of motion are (2.6-10) and (2.6-13). Equation (2.6-10) is identical to (2.6-2). Because $m = m_1 + m_2$, the model (2.6-13) reduces to the simpler model (2.6-1) when the arm's speed and acceleration are small ($\dot{\phi} \approx \ddot{\phi} \approx 0$). In this case, the reaction forces are small and can be neglected. as was done in deriving the simpler model.

Because (2.6-10) is identical to (2.6-2), the torque T required is the same as that shown in Figure 2.6-4. The motor force f is obtained by substituting $\ddot{x}$, $\dddot{\phi}$, $\ddot{\phi}$, $\dot{\phi}$ from (2.6-4), (2.6-8) and their derivatives into (2.6-13). The plot is labeled f_2 in Figure 2.6-4. This model predicts that the required force magnitude will be 283.5 N at $t = 4$. This is 8% greater than the force of 262.5 N predicted by the simpler model, but is still less than the available force of 300 N.

Thus the proposed design appears to be feasible so far. Further analysis would depend on the specifics of the intended application, and might include a study of the effects of allowing the hand to move while the arm is in motion. This would require that the equations of motion be derived for the case where r_2 is a specified function of time.

Matlab Simulation Example:
Solving Systems of Equations

- From Matlab website: http://www.mathworks.com/support/tech-notes/1500/1510.html#time

Consider the second-order system

$$u''(x) + e^x v'(x) + 3u(x) = e^{2x}$$
$$v''(x) + \cos(x)u'(x) + u(x) = \sin(x)$$

First reduce this system of second-order ODEs to a first-order differential equation by introducing the vector

$$\mathbf{y} = \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix}$$

Next, rewrite the above system of equations as

$$\frac{d}{dx}\begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 0 & -e^x \\ 0 & 0 & 0 & 1 \\ -1 & -\cos(x) & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} + \begin{pmatrix} 0 \\ e^{2x} \\ 0 \\ \sin(x) \end{pmatrix}$$

Matlab Simulation Example:
Solving Systems of Equations

- From Matlab website: http://www.mathworks.com/support/tech-notes/1500/1510.html#time

Enter this into MATLAB in the following format:

```
function dy = secondode(x,y)
% function to be integrated
dy = zeros(4,1);



dy(1) = y(2);
dy(2) = -3*y(1) -exp(x)*y(4) + exp(2*x);
dy(3) = y(4);
dy(4) = -y(1) -cos(x)*y(2) + sin(x);
```

Note the change of variable from x to t (it is simply the independent variable).

Now solve the system using ODE45 and the initial conditions $u(0) = 1$, $u'(0) = 2$, $v(0) = 3$, $v'(0) = 4$ over the interval from $x = 0$ to $x = 3$. The commands you will need to use are:

```
xspan = [0 3];
y0 = [1; 2; 3; 4];
[x, y] = ode45(@secondode, xspan, y0);
```

The values in the first column of y correspond to the values of u for the x values in x. The values in the second column of y correspond to the values of u', and so on.

- More on ode45 commands: http://www.mathworks.com/access/helpdesk/help/techdoc/ref/ode45.html

Homework 3: chapter 3

- 3.14, 3.21,3.24,3.25,3.31,3.33,3.36
- Case study simulation:
  - Simulate the detailed model for 10 seconds with initial conditions $x_0 = 0, \dot{x}_0 = 0, \phi_0 = 10^0, \dot{\phi}_0 = 0$ and for three cases:
    1. T = 0, f =0
    2. T=370 Nm, f = 0
    3. T=0, f = 263 N
  - For each case, plot state vector versus time
  - Describe the behavior of the system for each case and discuss the stability of the system

Case Study Simulation Model

$$I_p \ddot{\phi} - mL\ddot{x}\cos\phi = T + mgl\sin\phi$$

or

$$a\ddot{\phi} + b\ddot{x} = c$$

where

$$a = I_p; \ b = -mL\cos\phi; c = T + mgl\sin\phi$$

$$(m+M)\ddot{x} - mL\cos\phi\ddot{\phi} + mL\sin\phi\dot{\phi}^2 = f$$

or

$$d\ddot{x} + e\ddot{\phi} = g$$

where $d = (m+M); \ e = -mL\cos\phi; \ g = -mL\sin\phi\dot{\phi}^2 + f$

Solve for $\ddot{x}$ *and* $\ddot{\phi}$

$$\ddot{x} = \frac{ag - ec}{ad - eb} = \frac{I_p\left(-mL\sin\phi\dot{\phi}^2 + f\right) + mL\cos\phi(T + mgl\sin\phi)}{I_p(m+M) + mL\cos\phi(-mL\cos\phi)}$$

$$\ddot{\phi} = \frac{c}{a} - \frac{b}{a}\left[\frac{ag - ec}{ad - eb}\right] = \frac{T + mgl\sin\phi}{I_p} + \frac{mL\cos\phi}{I_p}\left[\frac{I_p\left(-mL\sin\phi\dot{\phi}^2 + f\right) + mL\cos\phi(T + mgl\sin\phi)}{I_p(m+M) + mL\cos\phi(-mL\cos\phi)}\right]$$

```
tspan = [0 3];  % simulation time vector


% State vector = y = [x ẋ φ φ̇]'
y0 = [0 0 10* pi /180 0]';
[t, y] = ode45(@ secondode, tspan, y0);


x = y(:,1);  xdot = y(:,2);  phi = y(:,3);  phidot = y(:,4);
% Plot state vector against time
figure
subplot(221)
plot(t, x)
ylabel('x,[m]')
xlabel('t,[s]')

subplot(222)
plot(t, xdot)
ylabel('xdot,[m / s]')
xlabel('t,[s]')

subplot(223)
plot(t, phi)
ylabel('phi,[m]')
xlabel('t,[s]')

subplot(224)
plot(t, phidot)
ylabel('phidot,[rad / s]')
xlabel('t,[s]')
```

```
% function  to be integrated
function  dy = secondode(t, y)
dy = zeros(4,1);


% Define  parameters
I_p = 47.5;  % kg − m^2
L = 7 / 8;  % m
m = 40;  % kg
M = 100;  % kg
g = 9.81;  % m / s^2
T = 0;
f = 0;
a = I_p;
b = −m* L* cos(y(3));
c = T + m* g * l * sin(y(3));
d = (m + M );
e = −m* L* cos(y(3));
g = −m* L* sin(y(3))* y(4)* y(4) + f ;


% State  vector  = y = [x ẋ φ φ̇]'= [y(1) y(2) y(3) y(4)]'
dy(1) = y(2);
dy(2) = (a* g − e* c) /(a* d − e* b);
dy(3) = y(4);
dy(4) = c / a − (b / a)* (a* g − e* c) /(a* d − e* b);
```

References

- Woods, R. L., and Lawrence, K., Modeling and Simulation of Dynamic Systems, Prentice Hall, 1997.

- Williams, J. H., Fundamentals of Applied Mechanics, Wiley, 1996

- Close, C. M., Frederick, D. H., Newell, J. C., Modeling and Analysis of Dynamic Systems, Third Edition, Wiley, 2002

- Lecture notes: *web.njit.edu/~gary/430/*

- Palm, W. J., Modeling, Analysis, and Control of Dynamic Systems