Modeling Mechanical Systems

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ME584
Agenda

• Idealized Modeling Elements
• Modeling Method and Examples
• Lagrange’s Equation
• Case study: Feasibility Study of a Mobile Robot Design
• Matlab Simulation Example

• Active learning: Pair-share exercises, case study
Idealized Modeling Elements
Inductive storage

- Electrical inductance
- Translational spring
- Rotational spring
- Fluid inertia
Capacitive Storage

Electrical capacitance

Translational mass

Rotational mass
Energy dissipators

- Electrical resistance
  - **Diagram:** A simple electrical circuit with a resistor labeled $R$, showing voltage $v_2$ and $v_1$, and current $i$.

- Translational damper
  - **Diagram:** A translational damper with force $F$, velocity $v_2$, and velocity $v_1$, labeled with a damping coefficient $b$.

- Rotational damper
  - **Diagram:** A rotational damper with torque $T$, angular velocity $\omega_2$, and angular velocity $\omega_1$, also labeled with a damping coefficient $b$. 
Springs

- Stiffness Element
- Stores potential energy

\[ f_s = K(x_2 - x_1) \]

Reality
- 1/3 of the spring mass may be considered into the lumped model.
- In large displacement operation springs are *nonlinear*.

Idealization
- Massless
- No Damping
- Linear

[Diagram showing linear and nonlinear springs]
Actual Spring Behavior

\[
\text{Restoring force} = \left( K + \mu \Delta x^2 \right) \Delta x
\]

1. Small motions for isolation \( \approx K \)
2. Large motions for static loads \( = K + \mu \Delta x^2 \)
Spring Connections

• Spring in series: $K_{EQ} = \frac{K_1 K_2}{K_1 + K_2}$

• Spring in parallel: $K_{EQ} = K_1 + K_2$
Dampers and Mass

- **Friction Element**

\[ f_D = B (\dot{x}_2 - \dot{x}_1) = B (v_2 - v_1) \]

- **Dissipate Energy**

- **Inertia Element**

\[ M \ddot{x} = \sum_i f_i = f_1 - f_2 - f_3 \]

- **Stores Kinetic Energy**
Dampers Connections

- **Dampers in series:** $B_{EQ} = \frac{B_1 B_2}{B_1 + B_2}$

- **Dampers in parallel:** $B_{EQ} = B_1 + B_2$
Modeling Mechanical Systems
Modeling Methods

- State assumptions and their rationales
- Establish inertial coordinate system
- Identify and isolate discrete system elements (springs, dampers, masses)
- Determine the minimum number of variables needed to uniquely define the configuration of system (subtract constraints from number of equations)
- Free body diagram for each element
- Write equations relating loading to deformation in system elements
- Apply Newton’s 2nd Law:
  - \( F = ma \) for translation motion
  - \( T = I\alpha \) for rotational motion
Example 1: Automobile Shock Absorber

Spring-mass-damper

\[ M \frac{d^2 y(t)}{dt^2} + b \frac{dy(t)}{dt} + ky(t) = r(t) \]

Free-body diagram

Wall friction, \( b \)

Force

M

(a)

(b)
Example 2: Mechanical System

- Draw a free body diagram, showing all forces and their directions
- Write equation of motion and derive transfer function of response $x$ to input $u$
Example 2: Mechanical System

a. - b. Free body:

\[ m \quad \begin{align*}
\Sigma F_x &= m \ddot{x} \\
-k(x-u) - b(\dot{x}-\dot{u}) &= m \ddot{x}
\end{align*} \]

b. Equation of: Using Newton's second law:

\[ m \ddot{x} + b \dot{x} + kx = + b \dot{u} + ku \]

In $D$-operator notation:

\[ [mD^2 + bD + k]x = [bD + k]u \]

The transfer function is:

\[ \frac{x}{u} = \frac{bD + k}{mD^2 + bD + k} \]
Example 3: Two-Mass System

- Derive the equation of motion for $x_2$ as a function of $F_a$. The indicated damping is viscous.
Example 3: Two-Mass System

Equations of motion for the two masses:

\[ F_a + k_2 (x_2 - x_1) - b_1 \dot{x}_1 = m_1 \ddot{x}_1 \]

\[-k_2 (x_2 - x_1) - k_2 x_2 - b_2 \dot{x}_2 = m_2 \ddot{x}_2 \]

Or

\[ m_1 \ddot{x}_1 + b_1 \dot{x}_1 + k_2 x_1 - k_2 x_2 = F_a \]

\[[m_1 D^2 + b_1 D + k_2] x_1 - k_2 x_2 = F_a \]

and

\[ m_2 \ddot{x}_2 + b_2 \dot{x}_2 + 2k_2 x_2 - k_2 x_1 = 0 \]

\[-k_2 x_1 + [m_2 D^2 + b_2 D + 2k_2] x_2 = 0 \]

Use Cramer's rule to solve for \( x_2 \)

\[ x_2 = \frac{k_2 F_a}{[m_1 D^2 + b_1 D + k_2][m_2 D^2 + b_2 D + 2k_2] - k_2^2} \]
Example 4: Three-Mass System

- Draw the free-body-diagram for each mass and write the differential equations describing the system
Example 4: Three-Mass System

\[ M_1 \ddot{x}_1 + (B_1 + B_2)\dot{x}_1 + (K_1 + K_2)x_1 - B_2\dot{x}_2 - K_2x_2 = 0 \]
\[ -B_2\dot{x}_1 - K_2x_1 + M_2\ddot{x}_2 + (B_2 + B_3)\dot{x}_2 + (K_2 + K_3)x_2 - B_3\dot{x}_3 - K_3x_3 = 0 \]
\[ -B_3\dot{x}_2 - K_3x_2 + M_3\ddot{x}_3 + B_3\dot{x}_3 + K_3x_3 = f_a(t) \]
Example 5: Pair-Share Exercise

- All springs are identical with constant $K$
- Spring forces are zero when $x_1=x_2=x_3=0$
- Draw FBDs and write equations of motion
- Determine the constant elongation of each spring caused by gravitational forces when the masses are stationary in a position of static equilibrium and when $f_a(t) = 0$. 
Example 5: Pair-Share Exercise:

(a) Summing the forces shown on each of the free-body diagrams and collecting terms, we obtain

\[ M_1 \ddot{x}_1 + B_1 \dot{x}_1 + 2Kx_1 - Kx_3 = M_1 g \]
\[ M_2 \ddot{x}_2 + B_2 \dot{x}_2 + 2Kx_2 - B_2 \dot{x}_3 - Kx_3 = M_2 g \]
\[ -Kx_1 - B_2 \dot{x}_2 - Kx_2 + M_3 \ddot{x}_3 + B_2 \dot{x}_3 + 2Kx_3 = M_3 g + f_a(t) \]

(b) Letting \( f_a(t) = 0 \), replacing \( x_1, x_2, \) and \( x_3 \) by the constant displacements \( x_{1_0}, x_{2_0}, \) and \( x_{3_0} \), and noting that all the derivatives of these constant displacements are zero, we have the following three algebraic equations.

\[ 2x_{1_0} - x_{3_0} = \frac{M_1 g}{K}, \quad 2x_{2_0} - x_{3_0} = \frac{M_2 g}{K}, \]

and \[ -x_{1_0} - x_{2_0} + 2x_{3_0} = \frac{M_3 g}{K} \]

Solving these equations simultaneously gives

\[ x_{1_0} = (3M_1 + M_2 + 2M_3) \frac{g}{4K} \]
\[ x_{2_0} = (M_1 + 3M_2 + 2M_3) \frac{g}{4K} \]
\[ x_{3_0} = (M_1 + M_2 + 2M_3) \frac{g}{2K} \]
Example 5: Pair-Share Exercise:

The four spring elongations are $x_{10}, x_{20}$, and

\[
\begin{align*}
    x_{30} - x_{10} &= (-M_1 + M_2 + 2M_3) \frac{g}{4K} \\
    x_{30} - x_{20} &= (M_1 - M_2 + 2M_3) \frac{g}{4K}
\end{align*}
\]

Note that the elongations are not affected by the viscous damping coefficients $B_1$ and $B_2$. 
Example 6: Pair-Share Exercise

- Assume that the pulley is ideal
  - No mass and no friction
  - No slippage between cable and surface of cylinder (i.e., both move with same velocity)
  - Cable is in tension but does not stretch
- Draw FBDs and write equations of motion
- If pulley is not ideal, discuss modeling modifications
Example 6: Pair-Share Exercise

When drawing the free-body diagrams, note that the downward force of the cable on \( M_2 \) is the same as the force of the cable to the right on \( M_1 \) because of the pulley. Summing the forces shown on each of the diagrams and collecting terms, we get:

\[
M_1 \ddot{x}_1 + B_1 \dot{x}_1 + K_1 x_1 - K_1 x_2 = f_a(t)
\]
\[
-K_1 x_1 + M_2 \ddot{x}_2 + B_2 \dot{x}_2 + (K_1 + K_2)x_2 = M_2 g
\]

- Pulley is not ideal
  - Add rotation mass and friction
  - Model the slippage behaviors
  - Add spring to model cable
**Example 7: Electric Motor**

- An electric motor is attached to a load inertia through a flexible shaft as shown. Develop a model and associated differential equations (in classical and state space forms) describing the motion of the two disks $J_1$ and $J_2$.
- Torsional stiffness is given in Appendix B
Example 7: Electric Motor

The torsional stiffness of the shaft is given by:

\[ k_r = \frac{GJ}{L} \]

And torque in the shaft by:

\[ k_T (\theta_2 - \theta_1) \]

For disk 1:

\[ \sum M_x = J_1 \ddot{\theta}_1 \]

\[ -T(i) + k_T (\theta_2 - \theta_1) = J_1 \ddot{\theta}_1 \]
Example 7: Electric Motor

or \[ J_1 \ddot{\theta}_1 + k_T \dot{\theta}_1 - k_T \theta_2 = -T(t) \]

For disk 2 \[ \sum M_x = J_2 \ddot{\theta}_2 \quad T_L - k_T (\theta_2 - \theta_1) = J_2 \ddot{\theta}_2 \]

or \[ J_2 \ddot{\theta}_2 + k_T \dot{\theta}_2 - k_T \theta_1 = T_L \]

Classical form: Convert the two second-order equations into a single fourth-order equation. Using the \( D \)-operator notation:

From the second equation:

\[ \theta_2 = \frac{T_L + k_T \theta_1}{J_2 D^2 + k_T} \]

Substitute into the first:

\[ [J_2 D^2 + k_T][J_1 D^2 + k_T] \theta_1 - k_T^2 \theta_1 = -[J_2 D^2 + k_T]T(t) + k_T T_L \]
Example 7: Electric Motor

For the state-space form, let:

\[
\begin{align*}
  x_1 &= \dot{\theta}_1 \\
  \dot{x}_1 &= \ddot{\theta}_1 \\
  x_2 &= \theta_1 \\
  \dot{x}_2 &= \dot{\theta}_1 = x_1 \\
  x_3 &= \dot{\theta}_2 \\
  \dot{x}_3 &= \ddot{\theta}_2 \\
  x_4 &= \theta_2 \\
  \dot{x}_4 &= \dot{\theta}_2 = x_3
\end{align*}
\]

Substituting gives:

\[
\begin{align*}
  \dot{x}_1 &= \left[ -T(t) - k_T x_2 + k_T x_4 \right] \\
  \dot{x}_2 &= x_1 \\
  \dot{x}_3 &= \left( T_L - k_T x_4 + k_T x_2 \right) \\
  \dot{x}_4 &= x_3
\end{align*}
\]
Example 8: Pair-Share Exercise: Copy Machine

- The device from a copying machine is shown. It moves in a horizontal plane. Develop the dynamic model, assuming that mass of bar is negligible compared to attached mass $m_2$ and angular motions are small. The mass is subjected to a step input $F$, find an expression for the displacement of point B after the transient motions have died out.
Example 8: Pair-Share Exercise: Copy Machine

For a massless bar:
\[ \sum M_0 = J \ddot{\theta} = 0 \quad -k_1 a^2 \theta - k_2 L (L \theta - y) = 0 \]

For the mass:
\[ \sum F_y = m_2 \ddot{y} \quad F + k_2 (L \theta - y) - b \dot{y} = m_2 \ddot{y} \]

From the first equation:
\[ \theta = \frac{k_2 Ly}{(k_1 a^2 + k_2 L^2)} \]

Substituting into the second:
\[ m_2 \ddot{y} + b \dot{y} + k_2 y - \frac{k_2^2 L^2}{(k_1 a^2 + k_2 L^2)} y = F \]
Example 9: Mass-Pulley System

- A mechanical system with a rotating wheel of mass $m_w$ (uniform mass distribution). Springs and dampers are connected to wheel using a flexible cable without skip on wheel.
- Write all the modeling equations for translational and rotational motion, and derive the translational motion of $x$ as a function of input motion $u$.
- Find expression for natural frequency and damping ratio.
Example 9: Mass-Pulley System

For the mass: 
\[ \sum F_x = m_L \ddot{x} \quad f - k_2 x - b \dot{x} = m_L \ddot{x} \]

For the pulley: 
\[ \sum M_o = J_o \ddot{\theta} \quad \frac{d}{2} k_1 (u - \frac{d}{2} \dot{\theta}) - f \frac{d}{2} = J_o \ddot{\theta} \]

Solving for the force \( f \): 
\[ f = - \left[ J_o \ddot{\theta} - \frac{d}{2} k_1 \left( u - \frac{d}{2} \dot{\theta} \right) \right] \]

Variables \( x \) & \( \theta \) are not independent:
\[ x = \frac{d}{2} \theta \quad \text{or} \quad \theta = 2 \left( \frac{x}{d} \right) \]
Example 9: Mass-Pulley System

Thus

\[ f = - \frac{J_0 \frac{2}{d} \ddot{x} - \frac{d}{2} k_1 u + \left(\frac{d}{2}\right)^2 k_1 \frac{2}{d} x}{\left(\frac{d}{2}\right)} \]

\[ f = - J_0 \left(\frac{2}{d}\right)^2 \ddot{x} - k_1 u + k_1 x \]

Substitute \( f \) into the equation of motion for the mass.

\[ - J_0 \left(\frac{2}{d}\right)^2 \ddot{x} + k_1 u - k_1 x - k_2 x - b \ddot{x} = M_L \ddot{x} \]

Or

\[ \left[ m_L + J_0 \left(\frac{2}{d}\right)^2 \right] \ddot{x} + b \ddot{x} + (k_1 + k_2) x = k_1 u \]
Example 9: Mass-Pulley System

One standard second-order system form is: \[ \ddot{\eta} + 2\zeta \omega_n \dot{\eta} + \omega_n^2 \eta = 0 \]

Thus

\[
\omega_n = \sqrt{\frac{k_1 + k_2}{m_L + J_0 \frac{4}{d^2}}} \quad \quad \quad 2\zeta \omega_n = \frac{b}{m_L + J_0 \frac{4}{d^2}}
\]

\[
\zeta = \left( \frac{1}{2} \frac{b}{m_L + J_0 \frac{4}{d^2}} \right)^{\frac{1}{2}} \sqrt{\frac{m_L + J_0 \frac{4}{d^2}}{k_1 + k_2}}
\]

\[
\zeta = \frac{b / 2}{\sqrt{(k_1 + k_2)(m_L + J_0 \frac{4}{d^2})}}
\]
Example 10: Pair-Share Exercise: Double Pendulum

- The disk shown in the figure rolls without slipping on a horizontal plane. Attached to the disk through a frictionless hinge is a massless pendulum of length L that carries another disk. The disk at the bottom of the pendulum cannot rotate relative to the pendulum arm.
- Draw free-body diagrams and derive equations of motion for this system.
Example 10: Pair-Share Exercise:

Using the free body of the wheel:

\[ \sum F_x = m_1 \ddot{x} \quad \implies \quad -f - P_x = m_1 \ddot{x} \]  
(a)

\[ \sum F_y = m_1 \ddot{y} \quad \implies \quad -N - P_y = 0 \]  
(b)

\[ \sum M_{cg1} = J_{cg1} \ddot{\phi} \quad \implies \quad -f r_x = J_{cg1} \ddot{\phi} \]  
(c)

Using the free body of the pendulum:

\[ \sum F_x = m_2 \ddot{x} \quad \implies \quad P_x = m_2 \ddot{x} \]  
(d)

\[ \sum F_y = m_2 \ddot{y} \quad \implies \quad P_y - m_2 g = m_2 \ddot{y} \]  
(e)

\[ \sum M_{cg2} = J_{cg2} \ddot{\theta} \quad \implies \quad P_x L \cos \theta - P_y L \sin \theta = J_{cg2} \ddot{\theta} \]  
(f)
Example 10: Pair-Share Exercise: Double Pendulum

The position of the center of mass of the pendulum is given by

\[ \bar{p} = (-r\phi - L\sin\theta)i - (L\cos\theta)j \]

Thus the acceleration components of the mass center of the pendulum are:

\[ \ddot{x} = -r\ddot{\phi} - L\ddot{\theta}\cos\theta + L\dot{\theta}^2\sin\theta \quad \text{and} \quad \ddot{y} = L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta \]

From (d) and (e)

\[ P_x = m_2 (-r\ddot{\phi} - L\ddot{\theta}\cos\theta + L\dot{\theta}^2\sin\theta) \quad \text{and} \quad P_y = m_2 (L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta) \]

Substituting these results into (f) gives:
Example 10: Pair-Share Exercise:
Double Pendulum

\[ \ddot{\theta} + \frac{m_2 r L \cos \theta}{J_{cg2} + m_2 L^2} \ddot{\phi} + \frac{m_2 g L \sin \theta}{J_{cg2} + m_2 L^2} = 0 \]

For the free body of the wheel: \[ \ddot{x} = -r \dot{\phi} \]
Eqn. (a) becomes:

\[ f = -P_x + m_1 r \ddot{\phi} \]
Substitute into (c) to obtain:

\[ \ddot{\theta} + \frac{J_{cg1} + (m_1 + m_2) r^2}{m_2 r L \cos \theta} \ddot{\phi} - \dot{\theta}^2 \tan \theta = 0 \]