4.9 Equivalent resistance

\[ R_{eq} = \frac{R_2 \left( \frac{R_1 R_2}{R_1 + R_2} + \frac{R_3 R_4}{R_3 + R_4} \right)}{R_2 + \frac{R_1 R_2}{R_1 + R_2} + \frac{R_3 R_4}{R_3 + R_4}} \]

4.13 RLC circuit analysis

Component equations

\[ i_{R_1} = \frac{e_0 - e_1}{R_1} \]

\[ i_{R_2} = \frac{e_1}{R_2} \]

\[ i_L = \frac{e_1 - e_2}{L D} \quad \text{with } i_L(0) \]

\[ i_{C_2} = C_2 D e_2 \quad \text{with } e_2(0) \]

Node equations

\[ i_{R_1} = i_{R_2} + i_L \]

\[ i_L = i_{C_2} \]

Substitute component equations into node equations

\[ \frac{e_0 - e_1}{R_1} = \frac{e_1}{R_2} + \frac{e_1 - e_2}{L D} \quad \Rightarrow \quad \left[ \frac{L D}{R_2} + \frac{L D}{R_1} + 1 \right] e_1 = \frac{L D}{R_1} e_0 + e_2 \]

\[ \frac{e_1 - e_2}{L D} = C_2 D e_2 \quad \Rightarrow \quad \left[ L C_2 D^2 + 1 \right] e_2 = e_1 \]

Combine the above two equations
\[ \left[ L D \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + 1 \right] \left[ L C_2 D^2 + 1 \right] e_2 - e_2 = \frac{L D}{R_1} e_0 \]

Reduce

\[ \left[ L C_2 D^2 + \frac{R_1 R_2 C_2}{(R_1 + R_2)} D + 1 \right] e_2 = \left( \frac{R_2}{R_1 + R_2} \right) e_0 \]

The known initial conditions are \( i_L(0) \) and \( e_2(0) \); we need \( e_2(0) \) and \( \dot{e}_2(0) \).

\[ e_2(0) = \text{known} \]

\[ \dot{e}_2(0) = \frac{1}{C_2} i_{C_2}(0) = \frac{1}{C_2} i_L(0) \quad \text{(from component equation and node equation)} \]

natural frequency

\[ \omega_n = \frac{1}{\sqrt{L C_2}} \]

damping ratio

\[ \frac{2 \zeta}{\omega_n} = \frac{R_1 R_2 C_2}{(R_1 + R_2)} \quad \Rightarrow \quad \zeta = \frac{R_1 R_2}{2 (R_1 + R_2) \sqrt{\frac{L}{C_2}}} \]

static gain

\[ G_s = \frac{R_2}{R_1 + R_2} \]
4.16 *RLC circuit transfer function*

**Component equations**

\[ i_{R_1} = \frac{e_0 - e_1}{R_1} \]

\[ i_L = \frac{e_1 - e_2}{L\,D} \quad \text{with } i_L(0) \]

\[ i_{C_1} = C_1 \, D e_2 \quad \text{with } e_2(0) \]

\[ i_{R_2} = \frac{e_2 - e_3}{R_2} \]

\[ i_{C_2} = C_2 \, D e_3 \quad \text{with } e_3(0) \]

**Node equations**

\[ i_{R_1} = i_L, \quad i_L = i_{C_1} + i_{R_2}, \quad i_{R_2} = i_{C_1} \]

Substitute component equations into node equations.

\[ \frac{e_2 - e_1}{R_1} = \frac{e_1 - e_2}{L\,D} \quad \text{or} \quad \left[ \frac{L}{R_1} D + 1 \right] e_1 = \frac{L D}{R_1} e_0 + e_2 \]

\[ \frac{e_1 - e_2}{L\,D} = C_1 \, D e_2 + \frac{e_2 - e_1}{R_2} \quad \text{or} \quad \left[ \frac{L}{C_1} D^2 + \frac{L}{R_2} D + 1 \right] e_2 = \frac{L D}{R_2} e_3 + e_1 \]

\[ \frac{e_2 - e_3}{R_2} = C_2 \, D e_3 \quad \text{or} \quad \left[ R_2 C_2 D + 1 \right] e_3 = e_2 \]

Reduce to get \( e_3 \) as a function of \( e_0 \).

\[ \frac{e_3}{e_0} = \frac{1}{R_2 C_2 L C_1 D^3 + \left( L C_1 + L C_2 + R_1 C_2 R_2 C_1 \right) D^2 + \left( R_1 C_1 + R_2 C_2 + R C_3 \right) D + 1} \]
4.17 RLC circuit state-space derivation

Component equations

\[ i_{R_1} = \frac{e_0 - e_1}{R_1} \]

\[ i_L = \frac{e_1 - e_2}{L \cdot D} \text{ with } i_L(0) \]

\[ i_{C_1} = C_1 \cdot D \cdot e_2 \text{ with } e_2(0) \]

\[ i_{R_2} = \frac{e_2 - e_3}{R_2} \]

\[ i_{C_2} = C_2 \cdot D \cdot e_3 \text{ with } e_3(0) \]

Node equations

\[ i_{R_1} = i_L, \quad i_L = i_{C_1} + i_{R_1}, \quad i_{R_2} = i_{C_2} \]

From the component equations, we can select state variables as follows.

\[ u_1 = e_0 \]

\[ x_1 = i_L \quad \text{thus} \quad \dot{x}_1 = D i_L = \frac{e_1 - e_2}{L} = \frac{1}{L} e_1 - \frac{1}{L} x_2 \]

\[ x_2 = e_2 \quad \text{thus} \quad \dot{x}_2 = D e_2 = \frac{1}{C_1} i_{C_1} \]

\[ x_3 = e_3 \quad \text{thus} \quad \dot{x}_3 = D e_3 = \frac{1}{C_2} i_{C_2} \]

We need to eliminate \( e_1 \), \( i_{C_1} \), and \( i_{C_2} \). From the \( R_1 \) component equation and the first node equation,

\[ e_1 = e_0 - R_1 \cdot i_{R_1} = e_0 - R_1 \cdot i_L = u_1 - R_1 \cdot x_1 \]
From the second node equation and the $R_2$ component equation,

\[ i_{c_1} = i_L - i_{R_1} = i_L - \frac{e_2 - e_3}{R_2} = x_1 - \frac{1}{R_2} x_2 + \frac{1}{R_2} x_3 \]

From the last node equation and the $R_2$ component equation,

\[ i_{c_2} = i_{x_2} = \frac{e_2 - e_3}{R_2} = \frac{1}{R_2} x_2 - \frac{1}{R_2} x_3 \]

State-space representation

\[ \dot{x}_1 = -\frac{R}{L} x_1 - \frac{1}{L} x_2 + \frac{1}{L} u_1 \]

\[ \dot{x}_2 = \frac{1}{C} x_1 - \frac{1}{R_2 C_1} x_2 + \frac{1}{R_2 C_1} x_3 \]

\[ \dot{x}_3 = \frac{1}{R_2 C_2} x_2 - \frac{1}{R_2 C_2} x_3 \]

With the initial conditions

\[ x_1(0) = i_L(0) \]

\[ x_2(0) = e_2(0) \]

\[ x_3(0) = e_3(0) \]
4.25 Dual op-amp filter circuit

The two op-amp circuits can be treated independently since the output of each acts like a voltage source to the next stage. For the first op-amp, the two resistances $R$ are the same thus, the circuit is an inverter.

$$e_1 = -e_i$$

The second op-amp circuit is a low-pass filter.

$$e_2 = \frac{-R_f}{R_f C_f D + 1} e_i$$

The second op-amp drives the $RC$ circuit.

$$e_o = \frac{1}{R_i C_i D + 1} e_2$$

Combining,

$$e_o = \left[ \frac{1}{R_L C_L D + 1} \right] \left[ \frac{R_f}{R_i} \frac{R_f}{R_f C_f D + 1} \right] e_i = \frac{R_f}{R_L C_L R_f C_f D + 1 \left( R_L C_L + R_f C_f \right) D + 1}$$

The static gain is

$$G_s = \frac{R_f}{R_i}$$

Using $\tau_L = R_L C_L$ and $\tau_f = R_f C_f$
The natural frequency is

\[ \omega_n = \sqrt{\frac{1}{R_L C_L R_f C_f}} = \sqrt{\frac{1}{\tau_L \tau_f}} \]

The damping ratio is

\[ \zeta = \frac{(R_L C_L + R_f C_f)}{2 \sqrt{R_L C_L R_f C_f}} = \frac{(\tau_L + \tau_f)}{2 \sqrt{\tau_L \tau_f}} \]

Using the values stated,

\[ \frac{R_f}{R_i} = 1 \]

\[ \tau_L = 500 \text{ ohm} \times 10 \times 10^{-6} \frac{s}{\text{ohm}} = 0.005 \text{ s} \]

\[ \tau_f = 10 \times 10^3 \text{ ohm} \times 10^{-6} \frac{s}{\text{ohm}} = 0.010 \text{ s} \]

Thus, the static gain is

\[ G_s = 1 \]

The natural frequency is

\[ \omega_n = \frac{1}{\sqrt{0.005 \times 0.010 \text{ s}}} = 27.1 \frac{\text{rad}}{s} = 4.32 \text{ Hz} \]

The damping ratio is

\[ \zeta = \frac{(0.005 + 0.010)}{2 \sqrt{0.005 \times 0.010}} = 0.204 \]

4.26 Dual op-amp filter state-space derivation

From the analysis in Problem 4.25, the modeling equations are

\[ e_2 = \frac{G_s}{\tau_f} \frac{1}{D + 1} e_i \quad \text{with} \quad e_2(0) \]

\[ e_o = \frac{1}{\tau_L} \frac{1}{D + 1} e_2 \quad \text{with} \quad e_o(0) \]

where \( G_s = \frac{R_f}{R_i} \), \( \tau_L = R_L C_L \) and \( \tau_f = R_f C_f \)

The state-space representation for this system is

\[ u_1 = e_i \]
\[ x_1 = e_2 \quad \text{thus} \quad \dot{x}_1 = D e_2 = \frac{-x_1 + G_s u_i}{\tau_f} \]
\[ x_2 = e_o \quad \text{thus} \quad \dot{x}_2 = D e_o = \frac{-x_2 + x_1}{\tau_L} \]

4.30 Voltmeter measurement degradation

The transfer function of the original circuit is

\[ e_1 = \frac{1}{1 + \frac{10,000}{47,000}} e_o = 0.8246 e_o \]

The transfer function of the circuit with the voltmeter is

\[ e_1 = \frac{1}{1 + \frac{10,000}{47,000} \left(1 + \frac{47,000}{R_{\text{meter}}} \right)} e_o \]

If \( R_{\text{meter}} = 100 \text{ k}\Omega \), then voltage reading will be degraded by 7.6% (i.e., volt reading = 92.4% of undisturbed voltage).

If \( R_{\text{meter}} = 1 \text{ M}\Omega \), then voltage reading will be degraded by 0.8%.

Case Simulation Study: