Linearization and Review of Stability

ME584
Fall 2010
Lecture Objectives and Activities

• Review of stability
• Importance of linearization
• Linearization of nonlinear systems
  – Pair-share problems
A stable system is a system with a bounded response to a bounded input. Response to a displacement/initial condition will produce either a decreasing, neutral, or increasing response.
Stability Analysis – 1st Order ODE

1st - Order: $a_1 \frac{dx}{dt} + a_0 x = b_0 u$

Characteristic equation:

$$a_1 \lambda + a_0 = 0 \Rightarrow \lambda = -a_0 / a_1$$

System is stable if $\lambda < 0$, unstable if $\lambda > 0$

Example: $6\dot{x} + 2x = 2u$; $u = 0, x_0 = 1$

Characteristic equation: $6\lambda + 2 = 0 \Rightarrow \lambda = -3$

$x = x_o e^{-t/3}$

Example: $6\dot{x} - 2x = 2u$; $u = 0, x_0 = 1$

Characteristic equation: $6\lambda - 2 = 0 \Rightarrow \lambda = 3$

$x = x_o e^{t/3}$
Stability Analysis – 2\textsuperscript{nd} Order ODE

\[ 2^{\text{nd}} - \text{Order} : a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = b_0 u \]

Characterisitic Equation:

\[ \frac{1}{\omega_n^2} = \frac{a_2}{a_o}, \quad \frac{2\zeta}{\omega_n} = \frac{a_1}{a_o} \]

\[ \lambda^2 + 2\zeta\omega_n \lambda + \omega_n^2 = 0 \]

\[ \lambda_{1,2} = -\zeta\omega_n \pm j\omega_n \sqrt{1-\zeta^2} \]

System is unstable if \( \lambda_1 \) and/or \( \lambda_2 > 0 \)

Stable response

\( \zeta < 1 \): Underdamped (Oscillation)

\( \zeta > 1 \): Overdamped (No oscilliation)

\( \zeta = 1 \): Critically damped (No oscilliation)

Relative stability: degree of stability

\[ \begin{align*}
    & (a) \quad R(s) = \frac{b}{s^2 + ax + b} \\
    & (b) \quad R(s) = \frac{9}{s^2 + 9s + 9} \\
    & (c) \quad R(s) = \frac{9}{s^2 + 2s + 9} \\
    & (d) \quad R(s) = \frac{9}{s^2 + 6s + 9} \\
    & (e) \quad R(s) = \frac{9}{s^2 + 9s + 9}
\end{align*} \]
Stability Analysis – State Space (SS)

State space format

\[
\dot{x} = Ax
\]

Let \( x = ke^{\lambda t} \), substitute

\[
\lambda ke^{\lambda t} = Ake^{\lambda t} \quad \text{or} \quad \lambda x = Ax
\]

\((\lambda I - A)x = 0\)

Non – trivial solution if

\[
\det(\lambda I - A) = 0
\]
Stability Analysis with SS - Example

\[
\frac{dx}{dt} = \begin{bmatrix} -\alpha & -\beta & 0 \\ \beta & -\gamma & 0 \\ \alpha & \gamma & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\]

The characteristic equation is then

\[
\det(\lambda \mathbf{I} - \mathbf{A}) = \det\left\{ \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -\alpha & -\beta & 0 \\ \beta & -\gamma & 0 \\ \alpha & \gamma & 0 \end{bmatrix} \right\}
\]

\[
= \det\begin{bmatrix} \lambda + \alpha & \beta & 0 \\ -\beta & \lambda + \gamma & 0 \\ -\alpha & -\gamma & \lambda \end{bmatrix}
\]

\[
= \lambda[(\lambda + \alpha)(\lambda + \gamma) + \beta^2]
\]

system is stable when \(\alpha + \gamma > 0\) and \(\alpha \gamma + \beta^2 > 0\).
Importance of linearization

- Dynamics Analysis
- Control and estimation systems design
 Importance of linearization

• Is nature linear or nonlinear? physical systems?
  – Many physical systems behave linearly within some range of variable, but become nonlinear as variables increase without limit
  – Possible to linearize nonlinear systems

• Example: Pendulum

\[ m\ddot{\theta} + K\dot{\theta} + \frac{mg \sin(\theta)}{L} = \tau \]

  For small \( \theta \), \( \sin(\theta) = \theta \)

\[ m\ddot{\theta} + K\dot{\theta} + (mg / L)\theta = \tau \]

• Tractable analysis with linear model
Importance of linearization

• Is this system stable? $m\ddot{\theta} + K\dot{\theta} + (mg/\ell)\theta = \tau$

• State space model:

  \[ \begin{aligned} 
  & x = \text{state variables} \quad x = [\theta \ \dot{\theta}] \\
  & \dot{x} = Ax + Bu \\
  \end{aligned} \]

  where \( A = \begin{bmatrix} 0 & 1 \\ -g/\ell & -K/m \end{bmatrix} \); \( B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \); \( u = \tau \)

• Control

  \[ u = -Gx \]

  Where \( G \) is the control matrix,

  \[ \dot{x} = Ax + Bu = Ax + B(-Gx) = (A - BG)x \]

  Choose \( G \) to achieve desired performance

• Estimation

  \[ u = -\hat{G}\hat{x} \]

  where \( \hat{x} \) is estimate of \( x \)
Linear Approximation

\[ \dot{x} = \frac{dx}{dt} = f(x, u) \]

\[ x = \bar{x} + x^* \]

\[ u = \bar{u} + u^* \]

\( \bar{x} \): equilibrium value of \( x \) about which linearization is taken
also called (steady state value/nominal value)

\( \bar{u} \): equilibrium value of \( u \) about which linearization is taken

\( x^* \): small perturbation or variation of \( x \)

\( u^* \): small perturbation or variation of \( u \)

To solve for \( \bar{x} \) and \( \bar{u} \),
set \( f(\bar{x}, \bar{u}) = 0 \)

\( \bar{x} \) and \( \bar{u} \) can also be provided from testing
Taylor’s Expansion

\[ \frac{dx}{dt} = 0 + \frac{dx^*}{dt} = f(\bar{x}, \bar{u}) + \left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}, u=\bar{u}} x^* + \left. \frac{\partial f}{\partial u} \right|_{x=\bar{x}, u=\bar{u}} u^* + H.O.T. \]

\[ \frac{dx^*}{dt} \approx Ax^* + Bu^* \]

where

\[ A = \left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}, u=\bar{u}} \quad \text{and} \quad B = \left. \frac{\partial f}{\partial u} \right|_{x=\bar{x}, u=\bar{u}} \]

A and B are called Jacobian matrices.
Step 1.
If \( \bar{x} \) and \( \bar{u} \) are not specified, set \( \dot{x} = 0 \) to solve for \( \bar{x} \) and \( \bar{u} \).
Define \( x_1, x_2, \ldots, x_n \) and \( u_1, u_2, \ldots, u_m \), and form \( f_1, f_2, \ldots, f_n \).

Step 2.
Solve for \( A \) and \( B \). Assume \( A \) is \((n \times n)\) and \( B \) is \((n \times m)\)

\[
A = \left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}, \ u=\bar{u}} = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}
\]

and

\[
B = \left. \frac{\partial f}{\partial u} \right|_{x=\bar{x}, \ u=\bar{u}} = \begin{bmatrix}
\frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\
\frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m}
\end{bmatrix}
\]

Step 3.
Form \( \frac{dx^*}{dt} \approx Ax^* + Bu^* \)
Example

\[ m\ddot{\theta} + K\dot{\theta} + \frac{mg\sin(\theta)}{L} = 0 \]

let \( x_1 = \theta, \ x_2 = \dot{\theta}, \ u = 0 \) (no control input)

\[ f = \begin{cases} f_1 \\ f_2 \end{cases} = \begin{cases} \dot{x}_1 \\ \dot{x}_2 \end{cases} = \begin{cases} x_2 \\ -\frac{k}{m}x_2 - \frac{g}{L}\sin x_1 \end{cases} \]
Step 1

Set $\dot{x} = \begin{cases} \dot{x}_1 \\ \dot{x}_2 \end{cases} = 0,$

$\dot{x}_1 = x_2 = 0 \Rightarrow \bar{x}_2 = 0$

$\dot{x}_2 = \frac{-k}{m} \bar{x}_2 - \frac{g}{L} \sin \bar{x}_1 = 0$

$\Rightarrow \bar{x}_1 = 0, \pi, 2\pi$

*For this example, let us consider $\bar{x}_1 = 0$*

$\bar{x} = \begin{cases} \bar{x}_1 \\ \bar{x}_2 \end{cases} = \begin{cases} 0 \\ 0 \end{cases}$
Step 2

\[ A = \left. \frac{\partial f}{\partial x} \right|_{x=x, u=u} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \bigg|_{\bar{x}, \bar{u}} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} \bigg|_{\bar{x}, \bar{u}} & \frac{\partial f_2}{\partial x_2} \bigg|_{\bar{x}, \bar{u}} \end{bmatrix} \]

\[ \frac{\partial f_1}{\partial x_1} \bigg|_{\bar{x}, \bar{u}} = \frac{\partial}{\partial x_1} (x_2) \bigg|_{\bar{x}, \bar{u}} = 0 \]

\[ \frac{\partial f_1}{\partial x_2} \bigg|_{\bar{x}, \bar{u}} = \frac{\partial}{\partial x_2} (x_2) \bigg|_{\bar{x}, \bar{u}} = 1 \]

\[ \frac{\partial f_2}{\partial x_1} \bigg|_{\bar{x}, \bar{u}} = \frac{\partial}{\partial x_1} \left( -\frac{k}{m} x_2 - \frac{g}{L} \sin x_1 \right) \bigg|_{\bar{x}, \bar{u}} = -\frac{g}{L} \cos x_1 \bigg|_{\bar{x}_1=0} = -\frac{g}{L} \]

\[ \frac{\partial f_2}{\partial x_2} \bigg|_{\bar{x}, \bar{u}} = \frac{\partial}{\partial x_2} \left( -\frac{k}{m} x_2 - \frac{g}{L} \sin x_1 \right) \bigg|_{\bar{x}, \bar{u}} = -\frac{k}{m} \]

\[ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k}{m} x_2 - \frac{g}{L} \sin x_1 \end{bmatrix} \]
Step 2 (continued)

\[ B = \frac{\partial f}{\partial u} \bigg|_{x=\bar{x}, u=\bar{u}} = 0 \]
Step 3

\[
\frac{dx^*}{dt} = \begin{bmatrix} \dot{x}_1^* \\ \dot{x}_2^* \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}
\]

\[
\dot{x}_1^* = x_2^*
\]

\[
\dot{x}_2^* = -\frac{g}{L} x_1^* - \frac{k}{m} x_2^*
\]

The same as \( \ddot{\theta} = -\frac{g}{L} \theta - \frac{k}{m} \dot{\theta} \)
Pair-Share Exercise

Linearize the system about the point where the mass compresses the spring by 1 m and the applied force $u = 0$,

$$m\ddot{x} = u + mg - k_1 x - k_2 x^3$$

where,

$m = 200 \text{ kg}$

$g = 10 \text{ m/s}^2$

$k_1 = 1000 \text{ N/m}$

$k_2 = 1000 \text{ N/m}^3$
Step 1

Let $x_1 = x$ and $x_2 = \dot{x}$

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{u}{m} + g - \frac{k_1}{m} x_1 - \frac{k_2}{m} x_1^3 \\ x_2 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\bar{u} = 0$$
Step 2

\[ B = \begin{bmatrix} \frac{\partial^2 f}{\partial u^2} \\ \frac{\partial^2 f}{\partial x \partial u} \end{bmatrix} = \begin{bmatrix} 0.005 \end{bmatrix} \]

\[ A = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial u} & \frac{\partial^2 f}{\partial x_2 \partial u} \\ \frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u} \end{bmatrix} = \begin{bmatrix} \frac{k_1}{m} - \frac{3k_2}{x_1} & 0 \\ 0 & 20 \end{bmatrix} \]
Step 3

\[
\dot{x}^* = \begin{bmatrix} 0 & 1 \\ -20 & 0 \end{bmatrix} x^* + \begin{bmatrix} 0 \\ .005 \end{bmatrix} u^*
\]
A simple robot arm is modeled as

\[ I\ddot{\theta} = T - mgL\cos\theta \]

where \( I \) is moment of inertia of arm, \( m \) is the mass, and \( T \) is the torque that the motor supplies.

We want the motor to hold the arm at five angles:

\[ \theta_e = 0^\circ, 45^\circ, 90^\circ, 135^\circ, 180^\circ, 225^\circ \]

Find the torque required and determine what will happen if something hits the arm and slightly alters its position?
Step 1

Let \( x_1 = \theta, \ x_2 = \dot{\theta}, \ u = T \)

\[
f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} u - \frac{mgL \cos x_1}{I} \\ x_2 \end{bmatrix}
\]

\[
\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \theta_e \\ 0 \end{bmatrix}
\]

To find the torque \( \bar{u} \) at \( \bar{x} \), set \( x = \bar{x} \) and \( \dot{x}_1 = 0 \) and \( \dot{x}_2 = 0 \)

\[
\bar{u} = mgL \cos x_1
\]

\[
\bar{x}_1 = 0^\circ, \quad \bar{u} = mgL
\]

\[
\bar{x}_1 = 45^\circ, \quad \bar{u} = 0.707mgL
\]

\[
\bar{x}_1 = 90^\circ, \quad \bar{u} = 0
\]

\[
\bar{x}_1 = 135^\circ, \quad \bar{u} = -0.707mgL
\]

\[
\bar{x}_1 = 225^\circ, \quad \bar{u} = -0.707mgL
\]
Step 2

\[
A = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} |_{\bar{x}, \bar{u}} & \frac{\partial f_1}{\partial x_2} |_{\bar{x}, \bar{u}} \\
\frac{\partial f_2}{\partial x_1} |_{\bar{x}, \bar{u}} & \frac{\partial f_2}{\partial x_2} |_{\bar{x}, \bar{u}}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
\frac{mgL}{I} \sin \bar{x}_1 & 0
\end{bmatrix}
\]

\[
B = \left[\frac{\partial f_1}{\partial u} |_{\bar{x}, \bar{u}} \frac{\partial f_2}{\partial u} |_{\bar{x}, \bar{u}}\right] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]
Step 3

\[
\dot{x}^* = \begin{bmatrix} \dot{x}_1^* \\ \dot{x}_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{mgL \sin \bar{x}_1}{I} \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u^*
\]

\[
\dot{x}_2^* = \frac{mgL}{I} (\sin \bar{x}_1) x_1^* + \frac{u^*}{I}
\]
When Arm is Hit

\[ \dot{x}_2^* = \ddot{x}_1^*, \text{so} \]

\[ \ddot{x}_1^* - \frac{mgL}{I} (\sin \bar{x}_1) x_1^* = \frac{u^*}{I} \]

**Characteristic equation**

\[ \lambda^2 - \frac{mgL}{I} (\sin \bar{x}_1) = 0 \]

has roots

\[ \lambda_{1,2} = \pm \sqrt{\frac{mgL}{I}} \sin \bar{x}_1 \]

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**For**

- \( \bar{x}_1 = 0^\circ, \lambda_{1,2} = 0,0 \) (neutrally stable)
- \( \bar{x}_1 = 45^\circ, \lambda_{1,2} = \pm \sqrt{\frac{.707mgL}{I}} \) (unstable)
- \( \bar{x}_1 = 90^\circ, \lambda_{1,2} = \pm \sqrt{\frac{mgL}{I}} \) (unstable)
- \( \bar{x}_1 = 135^\circ, \lambda_{1,2} = \pm 0,0 \) (neutrally stable)
- \( \bar{x}_1 = 180^\circ, \lambda_{1,2} = \pm j \sqrt{\frac{.707mgL}{I}} \) (undamped / neutrally stable)
- \( \bar{x}_1 = 225^\circ, \lambda_{1,2} = \pm j \sqrt{\frac{.707mgL}{I}} \) (undamped / neutrally stable)
Lecture Recap

• Many nonlinear systems behave linearly with small perturbation

• Linearization procedure
  – Establish equilibrium
  – Solve for A and B

• Analysis is tractable with linear models

• Next lecture: Stability analysis and simulation with Matlab
References


• Palm, W. J., Modeling, Analysis, and Control of Dynamic Systems